Vague size predicates

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Abstract. Vague predicates such as heap, tall, bald, near to, nicer than, etc. are characterized by their Sorites susceptibility and the existence of borderline cases. In attempting to develop a general theory, diverse vague predicates are often analyzed in rather broad natural language settings. Because this has proven to be rather difficult, the phenomenon of vagueness is studied here in a narrower and strictly formalized setting: Vague size predicates such as "roughly the same size" and "negligible in size with respect to" are studied within the framework of a formalized theory of parthood extended by size predicates. Such a restricted axiomatic framework has a number of advantages: (i) it is easier to analyze logical and semantical aspects of vagueness; (ii) it is possible to study the interrelations between precise quantitative facts and the interpretation of vague and qualitative predicates in a restricted class of models; (iii) it is possible to precisely analyze the logical properties of vague and qualitative size predicates which are fundamental to formalized deductive reasoning. All three aspects are critical for the representation and reasoning about many vague predicates as they are used in many sciences and in medicine.

Keywords: formal ontology, qualitative representation and reasoning, vagueness, supervaluation, modal logic

1. Introduction

In many data-driven sciences, including biology and medicine, there is a need to link qualitative and often vague predicates such as "has normal systolic blood pressure", "has high systolic blood pressure", "has elevated levels of cholesterol", etc. to numerical representations of particular qualities such as systolic blood pressures and cholesterol levels as they inhere in specific organisms at specific times (Smith et al., 2007; OBO Consortium, 2009; The OBI consortium, 2010). This is because:

– Qualitative and often vague predicates are fundamental to scientific/medical reasoning;
– The truth conditions for sentences with qualitative and vague predicates such as "x has normal systolic blood pressure" are often specified using numerical constraints.

Consider the following example. In medicine it is common to infer elevated risks of heart disease from high systolic blood pressures and/or elevated levels of cholesterol. The logic of such types of inferences can be analyzed by considering the following argument:

\[ \begin{align*}
(P1) & \quad \text{x has a high systolic blood pressure.} \\
(P2) & \quad \text{For all } x, \text{ if } x \text{ has a high systolic blood pressure, then } x \text{ has an elevated risk of heart disease.} \\
(C1) & \quad \text{Therefore, } x \text{ has an elevated risk of heart disease.}
\end{align*} \]

The following observations can be made: All the statements that constitute argument (1) contain at least one qualitative and vague predicate; The argument as a whole is an instance of modus ponens combined with the rule of universal instantiation; Premise (P1) is factual in nature; Premise (P2) explicates the assumption that the causal interrelations between blood pressures and heart diseases are law-like and can be expressed logically as a universally quantified implication. Among others, these observations seem to support the conclusion that the semantic and ontological analysis of qualitative and vague predicates such as "has normal systolic blood pressure" is to be performed in a formal framework that preserves (large parts) of classical logic. On this view, reasoning from (P1) and (P2) to (C1) is sound only if (P1) and (P2) are both true.

Consider premise (P1). Knowledge about facts that involve particular qualities such as particular systolic blood pressures or particular cholesterol levels is usually obtained by measurement and therefore is quantitative. Examples of specific numerical values that represent facts about the measurement of particular systolic blood pressures and particular cholesterol levels include respectively 140 millimeters of...
mercury (mm Hg) and 90 milligrams per deciliter (mg/dL). To determine the truth or falsity of a premise such as (P1) one has to solve the problem of linking precise quantitative facts like

(F1) that \( x \) has a systolic blood pressure of 140 mm Hg (expressed as \( x_m = 140 \) mm Hg) to the truth or the falsity of sentences such as "\( x \) has high systolic blood pressure".

In medical practice one attempts to solve these kinds of problems by means of rules of the sort displayed in Table 1. Intuitively, for normal human beings, systolic blood pressures between 90 and 130 mm Hg are interpreted as normal and systolic blood pressures of 140 mm Hg and higher are interpreted as high. Values in between are interpreted as borderline high. That is, in the underlying truth theory (which, in its rudimentary form is partly displayed in Table 1), the truth conditions for sentences that contain vague predicates are given in the form of Tarski's 'condition T': \( S \) is true if and only if \( P \) (Tarski, 1944). Here the variable \( S \) ranges over (names of) sentences in the object language (the language of practicing medicine) and the variable \( P \) ranges over translations of \( S \) into the meta-language in which the meaning of predicates of the object language is fixed in terms of quantitative facts about measurements and observations.

There is an obvious tension between the claim that predicates such as "has normal systolic blood pressure", etc. are qualitative and vague and their interpretation using crisp numerical intervals of the sort displayed in Table 1. In particular, if the sentence "\( x \) has normal systolic blood pressure" is true if and only if the blood pressure \( x_m \) of \( x \) is between 90 – 130 mm Hg, then the predicate "has normal systolic blood pressure" is not vague but crisp. This is because quantitative representations are based on measurements, and given the finite resolution of measurement devices, the right hand side of the definition is logically equivalent to a disjunction of finitely many crisp cases, e.g., \( x_m = 90 \) mm Hg or \( x_m = 91 \) mm Hg or \( x_m = 92 \) mm Hg, etc. Consequently, definitions of the form given in Table 1 are misleading by implying an artificially crisp interpretation. However, that predicates such as "has normal systolic blood pressure" are indeed vague, can be seen easily by recognizing that the specific ranges of blood pressures and cholesterol levels in Table 1 are, to a certain degree, arbitrary. There are no scientific reasons to prefer the interval [90, 130] for determining the truth value of the predicate "has normal systolic blood pressure" over the intervals [91, 131], [91, 129.5], etc. That is, all these intervals (and many others more) are equally well suited candidates for specifying the range of numerical values that make the sentence "\( x \) has normal systolic blood pressure" true (at least in the context of practicing medicine).

Although there may be reasons to ignore this kind of arbitrariness for some practical purposes, such aspects must be part of a semantic/ontological analysis which aims (a) to determine the referents of predicates such as "has normal systolic blood pressure" and (b) to justify the definitional choices of the underlying truth theory. That is, the fact that there is no scientific justification for assigning the truth value True to the sentence "\( x \) has normal systolic blood pressure" if the measured systolic blood pressure of \( x \) is 130 mm Hg and False if the measured blood pressure is 130.5 mm Hg must be reflected in the underlying truth theory. Similar points can be made for the other vague predicates in Table 1.

Now consider premise (P2) of argument (1) and its relationship to the numeric values in Table 1. Intuitively, the intended purpose of the tri-partition of ranges of systolic blood pressures in the table seems to be to explicate a distinction between unequivocally normal systolic blood pressures, unequivocally high systolic blood pressures, and intermediate boundary cases. It then seems to be the case that the intended logical representation of the causal premise (P2) is:

| \( x \) has normal systolic blood pressure" | \( 90 \leq x_m \leq 130 \) mm Hg |
| \( x \) has borderline high systolic blood pressure" | \( 131 \leq x_m \leq 140 \) mm Hg |
| \( x \) has high systolic blood pressure" | \( x_m > 140 \) mm Hg |

| \( x \) has optimal levels of cholesterol" | \( x_m < 100 \) mg/dL |
| \( x \) has borderline high levels of cholesterol" | \( 130 \leq x_m \leq 159 \) mg/dL |
| \( x \) has high levels of cholesterol" | \( 160 \leq x_m \leq 189 \) mg/dL |
| \( x \) has very high levels of cholesterol" | \( x_m > 190 \) mg/dL |

Table 1

Ranges of quantitative values that precisify the meaning of vague predicates in the context of practicing medicine.
For all \( x \), if unequivocally, \( x \) has high systolic blood pressure, then \( x \) has an elevated risk of heart disease. \( (2) \)

The antecedent of \( (P2)' \) is much stronger than the antecedent of \( (P2) \) in the sense that, intuitively, the semantic operator "unequivocally" restricts the vague predicate "has high systolic blood pressure" to its 'crisp core'. That is, according to Table 1, systolic blood pressures of 140 mm Hg and higher are unequivocally high and systolic blood pressures between 90 and 130 mm Hg are unequivocally normal. \( (P2)' \) then explicates that causal inferences from systolic blood pressures to elevated risks of heart disease are permitted only for unequivocally high systolic blood pressures. Of course, for argument (1) to remain valid, the premise \( (P1) \) needs to be revised in accordance to \( (P2)' \).

The discussion of argument (1) illustrates that an ontological and semantic analysis of medical/scientific diagnostic and reasoning that involves vague predicates needs to include:

(i) An analysis of the relationships between precise quantitative representations of measured/observed facts, the interpretation of vague qualitative predicates, and truth conditions for sentences that contain such predicates;

(ii) An analysis of the role of vague and qualitative predicates in logical representations of the causal laws that are exploited in reasoning processes.

In this paper the treatment of (ii) will focus on the analysis of the meaning of the word "unequivocal". It is assumed that in premises such as \( (P2)' \) one is permitted to represent causal relationships as universally quantified implications if the word "unequivocal" is understood properly.

Not all reasoning in the sciences and in medicine that involves vague predicates is based on logical representations of causal interrelations. Consider, for example, the binary predicate "has roughly the same systolic blood pressure as" or similar predicates whose names start with "has roughly the same ...". Such predicates are very common even in many scientific discourses. In fact, due to limitations of the accuracy of measurements combined with the semantic limitations on the mapping between quantitative values and qualitative predicates discussed above, the following seems to be the case: Whenever one says that, based on observations such as measurements, two objects (of non-microscopic scale) have the same qualities, one is likely to mean that the two objects have roughly the same qualities.

Consider the reasoning in the following argument:

\( (P3) \) \( x \) has a normal systolic blood pressure.
\( (P4) \) For all \( x \) and \( y \), if \( x \) has a normal systolic blood pressure and \( x \) has roughly the same systolic blood pressure as \( y \), then \( y \) has a normal systolic blood pressure.

\( (C2) \) Therefore, \( y \) has a normal systolic blood pressure. \( (3) \)

The reasoning in this argument is based on the logical (rather than causal) interrelations between the vague predicates "has normal systolic blood pressure" and "has roughly the same systolic blood pressure as". This is captured formally in premise \( (P4) \). Although quite intuitive at first sight, reasoning of the kind displayed in (3) potentially leads to logical contradictions which are known as Sorites-style paradoxes (Hyde, 1996). To formalize reasoning that avoids such kinds of paradoxes, a logical analysis of predicates such as "has roughly the same systolic blood pressure as" is critical. This motivates a third important aspect of the ontological and semantic analysis of vague qualitative predicates in medical/scientific diagnosis and reasoning:

(iii) The analysis of the logical interrelations among vague qualitative predicates to support sound reasoning and to avoid Sorites-style paradoxes and other kinds of contradictions.

Numerical representations of particular qualities such as blood pressures, cholesterol levels, etc. can be understood as volumes of \( n \)-dimensional regions in abstract \( n \)-dimensional conceptual spaces (Gärdenfors, 2000) or quality spaces (Gangemi et al., 2003). Consider, for example, a one-dimensional quality space in which units are interpreted as millimeters of mercury (mm Hg). A blood pressure of 120 millime-
ters of mercury (mm Hg) can be understood as the volume of the interval\(^1\) that extends from the origin of the chosen base of the space (0) to the value 120.

In Euclidean spaces, the qualities of length, area, and volume coincide with the mathematical notion of Lebesgue measures (Burk, 1997). That is, the length of the line between two points can be understood as the Lebesgue measure of an interval in a one-dimensional Euclidean quality space. Similarly, the area under the curve of an EEG can be understood as the Lebesgue measure of a region of \(\mathbb{R}^2\). The volume of a patient’s liver can be understood as the Lebesgue measure of a region of \(\mathbb{R}^3\). Similarly for all \(n\)-dimensional topologically regular regions in \(n\)-dimensional quality spaces with a Euclidean structure. On these assumptions, numerical representations of measurements of particular blood pressures and cholesterol levels can be understood as Lebesgue measures of regions (intervals) in Euclidean quality spaces in which units are interpreted as corresponding to millimeters of mercury (mm Hg), milligrams per deciliter (mg/dL), etc. \(^2\)

Consider the numerical constraints in Table 1. Given the understanding of numerical representations of particular qualities such as systolic blood pressures as Lebesgue measures of regions in quality spaces, vague predicates such as "has normal systolic blood pressure", "has roughly the same systolic blood pressure as", etc. can be interpreted as sets or intervals formed by real numbers that are understood as Lebesgue measures of regions in those quality spaces.

Predicates such as "has normal systolic blood pressure", "has roughly the same systolic blood pressure as", etc. are rather domain-specific. The aim of this paper is to present a formal analysis that addresses (i)-(iii) for a more general class of predicates – vague size predicates. Size predicates are formal and their logical properties are domain-independent like parthood and connectedness predicates. The focus on size predicates is based on two assumptions, both of which seem to be justified by the discussion above:

(1) Numerical representations of a large class of measurable qualities including blood pressures, cholesterol levels, etc. can be understood as Lebesgue measures of regions in Euclidean quality spaces.

(2) Size predicates are interpreted in terms of Lebesgue measures of regions in arbitrary Euclidean spaces. The logical and semantic properties of size predicates therefore generalize to predicates that are interpreted as more specific measurable qualities (blood pressures, cholesterol levels, etc.).

Due their formal and domain-independent character, the analysis of logical and semantic interrelations between size predicates provides generally applicable and logically sound foundations for linking precise quantitative facts to the interpretation of vague qualitative predicates and for performing deductive reasoning with such vague and qualitative predicates.

2. Related work

Prominent theoretical and computational approaches to formally link vague and qualitative predicates to crisp and numerically represented qualities and to perform (automated) reasoning, are often based on statistics, probability theory and/or fuzzy logic (Zadeh, 1975). Although quite successful in some areas such as data mining (Hirsh, 2008), there are at least four problematic points that justify the investigation of alternative approaches for semantic and ontological purposes.

Firstly, neither statistics nor probability theory nor fuzzy logic seem to do justice to the fact that the interpretations of many vague predicates that are used in medical and scientific practice have the following

\(^1\)The volume of an interval in a one-dimensional space is its length. See also footnotes 2, 3, and 5.

\(^2\)Although it is possible to apply the formal theory to measures that are more general than Lebesgue measures including measures in non-euclidean measure spaces, the discussion in this paper focusses on volumes of \(n\)-dimensional regions in \(n\)-dimensional Euclidean spaces, i.e., Lebesgue measures. The idea is to use the most straightforward and most obvious notion of measurement for interpreting size predicates and to focus the discussion on the logic of such predicates and the ways how their vagueness affects their semantics and reasoning.
properties: (a) they are specified as crisp and idealized ranges of numeric values as indicated in Table 1; (b) within a given range there is no gradation/differentiation between the various numeric values; and (c) to a certain degree, the boundaries of the intervals of numeric values are arbitrary and established by fiat as determined by the needs of medical and scientific practice.

Consider, again, the ranges of numeric values that make the vague predicates in Table 1 true according to interpretations determined by medical practice. On such an interpretation, there is no gradient that makes predicates such as "has normal systolic blood pressure" 'more true' for numeric values that are in the center of the interval compared to numeric values that are nearer to the boundaries. In the USA you can verify this easily by inspecting the latest lab report for your blood work. If your measured values fall anywhere in the listed intervals of normal ranges, then you are considered fine. Otherwise there is a mark on the report indicating a non-normality. Moreover, as pointed out above, the specific ranges of blood pressures and cholesterol levels in Table 1 are, to a certain degree, arbitrary in the sense that there are no scientific reasons to prefer the interval $[90, 130]$ over similar intervals. The fiat character of singling out specific crisp ranges for the interpretation of vague predicates is difficult to capture logically using statistics, probability theory and/or fuzzy logic.

Secondly, the interpretation of vague predicates is often context-dependent (van Deemter, 1995; Keefe, 2003). For example "has normal systolic blood pressure", "has optimal cholesterol level", etc. can be interpreted in different ways in different contexts. For patients at risk of heart disease (due to pre-conditions such as diabetes, family history, etc.) more narrow numeric ranges for "has normal systolic blood pressure" and "has optimal cholesterol level" may apply.

Thirdly, the logic of vagueness may not be appropriately captured by statistical, probabilistic or fuzzy-logic based frameworks (Elkan, 1994; Keefe and Smith, 1996; Fine, 1975). Fourthly, as indicated in argument (1), reasoning about vague predicates in medical practice and in many scientific contexts seems to be based to large degrees on classical logic. Similarly, from a semantic and ontological perspective it seems desirable to analyze vague qualitative predicates and their use in the sciences and medicine in a way that preserves as many principles of classical logic as possible (Guarino, 1998; Gangemi et al., 2003; Smith, 2003).

One possible alternative to statistical, probabilistic and fuzzy-logic-based approaches for the purpose of formally analyzing (a) the semantic interrelations between vague and qualitative predicates and crisp and numerically represented qualities and (b) the reasoning using vague predicates, is to use a logical framework that is based on supervaluation semantics (van Fraassen, 1966; Fine, 1975; Pinkal, 1995). Using supervaluation has a number of advantages: Firstly, supervaluation semantics quite naturally deals with multitudes of equally good ways of linking crisp and numerical values to vague and qualitative predicates (Fine, 1975); Secondly, supervaluation semantics can be extended to take into account context dependency (Keefe, 2003); Thirdly, supervaluation was developed as a tool for formalizing the logic of vague predicates (van Fraassen, 1966; Fine, 1975). Moreover, it is well recognized in the literature that supervaluation captures important formal aspects of vagueness (Keefe and Smith, 1996); Fourthly, supervaluation semantics preserves many of the principles of classical logic (Fine, 1975; Pinkal, 1995).

Unfortunately, supervaluation semantics is usually formalized in abstract model-theoretic frameworks e.g., (Fine, 1975; Pinkal, 1995; Bennett, 1998). Although a powerful tool for logical analysis, abstract model-theoretic semantics cannot be linked easily to specific numeric values as they are listed in Table 1 and as they are obtained by measurements in science and medicine. The supervaluation semantics presented here is based on numerical constraints between a certain class of measures of regions in quality spaces. In this semantic framework constraints of the sort listed in Table 1 can serve as interpretations of vague predicates such as "has normal systolic blood pressure" and "has optimal cholesterol level", etc.

The focus of this paper is on logical and semantic interrelations between crisp and vague size predicates. In the axiomatic theory such interrelations are formalized by combining a mereology (Simons, 1987) which is extended by crisp size predicates, with work from Order of Magnitude Reasoning, especially (Raiman, 1991; Dague, 1993b), and work on semantic theories of vagueness, especially (Fine, 1975; Pinkal, 1985; Bennett, 1998; Bennett et al., 2008). The formal theory is presented in a modal logic (Hughes and Cresswell, 2004), which is similar to (Bennett, 1998; Halpern, 2004; Bennett et al., 2008). However
the general-purpose model-theoretic semantics is replaced by a semantics that is based on a particular class of parametrized linear constraints that was originally used by Raiman (1991) for the formalization of Order of Magnitude reasoning in qualitative physics. (See also (Raiman, 1991; Mavrovouniotis and Stephanopoulos, 1988; Dague, 1993a,b; Clementini et al., 1997; Davis, 1999).) Every constraint has a parameter that can take a value between 0 and 1 and every choice of a particular parameter value specifies a ‘possible world’ in the underlying semantics of the modal logic. This paper extends the literature on Order of Magnitude reasoning by giving an axiomatization on a mereological basis which includes an explicit supervaluation-based treatment of vagueness.

In its treatment of size predicates the presented formal theory goes beyond existing formalizations of qualitative size predicates, including (Gerevini and Renz, 1998; Cohn et al., 1997; Cohn and Hazarika, 2001; Schmidtke and Woo, 2007). It provides a more clear distinction between the understanding of size of a region as a volume-based measure from diameter-based conceptualizations (e.g., (Schmidtke and Woo, 2007)). In addition, the focus here is also on vagueness and not only the qualitative character of such predicates. Incorporating concepts from Order of Magnitude Reasoning facilitates the formalization of granularity and scale in the underlying mereology and size-based framework (Bittner and Donnelly, 2006). Applications of the presented formal theory to the study of vagueness in the classification of ecological and climatic regions and other geographic features such as bays can be found in (Bittner, 2011b) and (Feng and Bittner, 2010). The presented formal theory was incorporated in a vague mereogeometry in (Bittner, 2009).

To simplify the presentation, the formal theory of vague size predicates is developed in a logical framework that ignores the features of context dependency and higher order vagueness. An extended version of the formal theory that takes these aspects into account can be found in (Bittner, 2011a).

3. Modeling vagueness in a modal framework

Vagueness here is understood as a semantic phenomenon (as opposed to an epistemic (Williamson, 1999) or metaphysical (Tye, 1990) phenomenon) and is modeled within a framework that is based on supervaluation (Fine, 1975; Pinkal, 1985). To specify a semantics for vague predicates in the object language of an axiomatic theory, \( \mathcal{L}_V \), a first-order modal logic with identity is used. The syntax and semantics of \( \mathcal{L}_V \) are defined in the standard ways based on (Hughes and Cresswell, 2004).

3.1. Syntax and semantics

The language \( \mathcal{L}_V \) includes a set of variable symbols \( \text{Var} \) (letters from the end of the alphabet \( x, x_1, \ldots, y, z, w \)), a set of constant symbols \( \text{Const} \) (letters from the beginning of the alphabet \( a, a_1, \ldots, b, c, d \)). A term is either a variable or a constant. \( \mathcal{L}_V \) also includes a set of predicate symbols, \( \text{Pred} \). If \( F \) is a \( n \)-ary predicate symbol in \( \text{Pred} \) and \( t_1, \ldots, t_n \) are terms then \( F \ t_1 \ldots t_n \) is a well-formed formula. Complex, non-modal formulas are formed in the usual ways. In addition, \( \mathcal{L}_V \) includes the modalities \( U \) and its dual \( S \), i.e., if \( \alpha \) is a well-formed formula of \( \mathcal{L}_V \), then so are \( U\alpha \) and \( S\alpha \). Following Bennett (1998), \( U\alpha \) is interpreted as \( \alpha \) is ‘unequivocally’ true, i.e., true on all precisifications. \( S\alpha \) is interpreted as \( \alpha \) is true on some precisification. Intuitively, precisifications represent ways of making the meaning of a vague predicate precise. That is, a vague predicate has different crisp interpretations in different ‘worlds’ or precisification points.

A \( \mathcal{L}_V \) model is a structure \( \langle D, \Omega, V \rangle \), where \( D \) is the non-empty domain of quantification and \( \Omega \) is a non-empty set of precisification points or precisification parameters. \( V \) is the interpretation function, which maps the members of \( \text{Const} \) to members of \( D \). If \( F \in \text{Pred} \) is an \( n \)-ary predicate then \( V(F) \) is a set of \( n + 1 \)-tuples of the form \( (d_1, \ldots, d_n, \omega) \) with \( d_1, \ldots, d_n \in D \) and \( \omega \in \Omega \). At all precisification points \( \omega \in \Omega \) (in all possible worlds), the variables of \( \mathcal{L}_V \) range over all the members of \( D \). The interpretation of constants is the same at all precisification points. A variable assignment \( \mu \) is a function such that for every
variable $x \in \text{Var}$, $\mu(x) \in \mathcal{D}$. The function $V_\mu(t)$ yields the value of $t \in \text{Var} \cup \text{Const}$ with respect to $V$ and $\mu$. $V_\mu(t)$ is defined as

$$
V_\mu(t) = V(t) \quad \text{iff} \quad t \in \text{Const}
$$

$$
V_\mu(t) = \mu(t) \quad \text{iff} \quad t \in \text{Var}
$$

Every well-formed formula has a truth value which is defined as follows:

$$
V_\mu(F t_1 \ldots t_n, \omega) = 1 \text{ if } V_\mu(t_1), \ldots, V_\mu(t_n), \omega) \in V(F) \text{ and 0 otherwise;}
$$

$$
V_\mu(\alpha \land \beta, \omega) = 1 \text{ if } V_\mu(\alpha, \omega) = 1 \text{ and } V_\mu(\beta, \omega) = 1 \text{ and 0 otherwise;}
$$

$$
V_\mu(\alpha \lor \beta, \omega) = 1 \text{ if } V_\mu(\alpha, \omega) = 1 \text{ or } V_\mu(\beta, \omega) = 1 \text{ and 0 otherwise;}
$$

$$
V_\mu(\neg \alpha, \omega) = 1 \text{ if } V_\mu(\alpha, \omega) = 0 \text{ and 0 otherwise;}
$$

$$
V_\mu(\alpha \rightarrow \beta, \omega) = 1 \text{ if } V_\mu(\alpha, \omega) = 0 \text{ or } V_\mu(\beta, \omega) = 1 \text{ and 0 otherwise;}
$$

$$
V_\mu((x)\alpha, \omega) = 1 \text{ if } V_\mu(\alpha, \omega) = 1 \text{ for every } x\text{-alternative } \rho \text{ of } \mu \text{ and 0 otherwise;}
$$

$$
V_\mu(U \alpha, \omega) = 1 \text{ if } V_\mu(\alpha, q) = 1 \text{ for all } q \in \Omega \text{ and 0 otherwise.}
$$

A well-formed formula $\alpha$ is true in $<\mathcal{D}, \Omega, V>$, i.e. $V_\mu(\alpha) = 1$, if and only if $V_\mu(\alpha, \omega) = 1$ for all $\omega \in \Omega$ and all assignments $\mu$. $\alpha$ is $LV$-valid if $\alpha$ is true in all $LV$ models. To simplify the presentation, the explicit distinction between $V$ and $\mu$ will be omitted. The context will make clear wether $V$ is the interpretation of a predicate symbol or the interpretation of a formula. Individual constants in the object language are written in *italics* and for corresponding domain members the Sans Serif font is used. Similarly for variables.

$LV$ includes the usual rules and axioms of first order logic with identity, a rule of necessitation for $U$, and the S5-axiom schemata $K_U$, $T_U$, and $5_U$ (Hughes and Cresswell, 2004). $S$ is defined in the usual way as the dual of $U$ ($D_S$). The Barcan formula and its converse are true in all $LV$-models ($BC_U$).

$$
D_S \quad S\alpha \equiv \neg U\neg \alpha
$$

$$
T_U \quad U\alpha \rightarrow \alpha
$$

$$
5_U \quad S\alpha \rightarrow U S\alpha
$$

$$
K_U \quad U(\alpha \rightarrow \beta) \rightarrow (U\alpha \rightarrow U\beta)
$$

$$
BC_U \quad (x)U\alpha \leftrightarrow U(x)\alpha
$$

### 3.2. Additional sentence operators

Additional sentence operators are introduced to characterize properties of vague predicates: $\alpha$ is *definite* if and only if either unequivocally $\neg \alpha$ or unequivocally $\alpha$ ($D_D$); $\alpha$ is *indefinite* if and only if on some precisification $\alpha$ and on some precisification $\neg \alpha$ ($D_I$).

$$
D_D \quad D\alpha \equiv U\neg \alpha \lor U\alpha
$$

$$
D_I \quad I\alpha \equiv S\alpha \land S\neg \alpha
$$

The operator $I$ corresponds to Fine’s indefiniteness operator (Fine, 1975) and the operator $D$ corresponds to Pinkal’s definiteness operator (Pinkal, 1995, 1985). One can prove that $\alpha$ is definite if and only if $\alpha$ is not indefinite ($T_{LV1}$) and that $\alpha$ is definite if and only if $\neg \alpha$ is definite ($T_{LV2}$).

$$
T_{LV1} \quad D\alpha \leftrightarrow \neg I\alpha
$$

$$
T_{LV2} \quad D\alpha \leftrightarrow D\neg \alpha
$$

### 3.3. Representing properties of vague predicates

Let $F$ be a n-ary predicate. Given the set of precisifications $\Omega$, there will be a set of n-tuples $<d_1, \ldots, d_n> \in \mathcal{D} \times \ldots \times \mathcal{D}$ such that $(d_1, \ldots, d_n, \omega) \in V(F)$ for all (no) choices of $\omega \in \Omega$. This set is called the positive (negative) extension, $\delta(F)^+$ ($\delta(F)^-$), of the predicate $F$:

$$
\delta(F)^+ = \{<d_1, \ldots, d_n> \in \mathcal{D} \times \ldots \times \mathcal{D} \mid \forall \omega \in \Omega : <d_1, \ldots, d_n, \omega> \in V(F)\}
$$

$$
\delta(F)^- = \{<d_1, \ldots, d_n> \in \mathcal{D} \times \ldots \times \mathcal{D} \mid \forall \omega \in \Omega : <d_1, \ldots, d_n, \omega> \notin V(F)\}
$$
Clearly, if \( F \) is crisp, then \( \delta(F)^+ \cup \delta(F)^- = D_1 \times \ldots \times D_n \). Of course, there is no reason to expect that the positive and negative extensions of vague predicates are crisp sets. This can be addressed by extending the presented framework using a multi-dimensional modal logic. Unfortunately, this goes beyond the scope of this paper and is developed in (Bittner, 2011a).

Vague predicates are characterized by the existence of borderline cases (Keefe and Smith, 1996).

**Definition 1.** The domain member \( d \in D \) is a boundary case for the unary predicate \( F \) if and only if \( d \) is neither a member of the positive extension of \( F \) nor a member of the negative extension of \( F \), i.e., \( d \not\in (\delta(F)^+ \cup \delta(F)^-) \). The ordered pair \( \langle d_1, d_2 \rangle \in D \times D \) is a boundary case for the binary predicate \( R \) if and only if \( \langle d_1, d_2 \rangle \not\in (\delta(R)^+ \cup \delta(R)^-) \). Similarly for \( n \)-ary predicates.

It is assumed here that the existence of boundary cases is a necessary and sufficient condition for distinguishing vague and crisp size predicates. That is, a size predicate \( F \) is vague if and only if \( F \) has boundary cases.

In the presented axiomatic theory the distinction between crisp and vague predicates is made explicit by including axioms that incorporate the sentence operators \( U \), \( S \), \( D \), and \( l \). For example, that the \( n \)-ary predicate \( F \) is crisp will be made explicit by an axiom of the form \( (x_1 \ldots x_n)D(F \ x_1 \ldots x_n) \). Similarly, that the \( n \)-ary predicate \( F' \) is vague will be made explicit by an axiom of the form \( (\exists x_1 \ldots x_n)l(F' \ x_1 \ldots x_n) \).

Axioms are included such that every predicate is either crisp or vague. Since for a given predicate \( F \) the property of crispness or vagneness is specified by an axiom or a theorem, it follows that \( F \) has the property of crispness or vagueness unequivocally. That is, it is assumed that whether a predicate is vague or not is not subject to vagueness.

4. A mereology with size predicates

The logic \( LV \) is used to present a mereology with size predicates. The language of \( LV \) is extended by the primitive predicate symbols \( P, \sim, \approx \) and by axioms constraining the interpretations of these symbols. In the presented theory axioms and theorems are, wherever possible, written as non-modal first order sentences. Leading universal quantifiers are generally omitted. Theorems (T1) - (T33) have been verified as provable from the axioms of \( LV \) and the axioms (A1)-(A30) using the theorem proving environment Isabelle (Paulson and Nipkow, 2005). The computational representation of the axiomatic theory can be accessed at [http://www.buffalo.edu/~bittner3/Theories/VagueSizePredicates](http://www.buffalo.edu/~bittner3/Theories/VagueSizePredicates). In the computational representation modal sentences are translated into first order sentences in the standard way (del Cerro and Herzig, 1995, p. 516). The consistency of the axiomatic theory is demonstrated in Section 7.

4.1. \( \Omega \)-structures

Models of the presented theory, \( \Omega \)-structures, are specific \( LV \) structures of the form

\[
\langle \Omega, D, \cup, \subseteq, ||\rangle. \tag{7}
\]

The members of \( D \) are the non-empty regular closed subsets of some set \( X \) (Alexandroff, 1961) that have a finite Lebesgue measure (Burk, 1997). The set \( X \) is either finite or identical to \( \mathbb{R}^n \). If \( X \) is \( \mathbb{R}^n \), then the members of \( D \) represent \( n \)-dimensional regions of \( n \)-dimensional space which do not have spikes or holes of lower dimension. The relation \( \subseteq \) is the subset relation among the members of \( D \). \( ||\) is a function from members of \( D \) to the positive real numbers such that \( ||d|| \) yields the Lebesgue measure of the set \( d \). The Lebesgue measure is a formalization of the notion of the length of \( d \) if \( d \) is a regular subset of \( \mathbb{R}^1 \). \( ||d|| \) is the area of \( d \) if \( d \) a regular subset of \( \mathbb{R}^2 \) and \( ||d|| \) is the volume of \( d \) if \( d \) a regular subset of \( \mathbb{R}^3 \). If \( X \) is finite, then the Lebesgue measure is a formalization of the notion of the cardinality of the subsets of \( X \). In
what follows the focus is on cases in which $X$ is $\mathbb{R}^n$. Relevant properties of $\|d\|$ are listed as (L1)-(L6) on page 11.\(^3\)

Given the numerical character of the models used in this paper, specific numerical constraints on $\Omega$ are necessary to ensure that the axioms are valid in $\Omega$-structures. (This is particularly the case for axioms that include modal operators.) These constraints will be discussed in Section 7. In practical applications in which the members of $\Omega$ determine admissible blood pressures, cholesterol levels, etc., it will often be convenient to assume that $\Omega$ is not just a set of real numbers, but a (topologically closed) interval that is subject to the numerical constraints of Section 7.

4.2. The mereology of regions

As the mereological basis the primitive binary predicate $P$ is introduced in the formal theory. Intuitively, $P_{xy}$ is interpreted as ‘the region $x$ is part of region $y$’. Formally, the parthood predicate $P$ is at all precisification points, $\omega \in \Omega$, interpreted as the subset relation, $\subseteq$, among the members of $D$:

$$V(P) =_{df} \{(d_1, d_2, \omega) \in D \times D \times \Omega \mid d_1 \subseteq d_2\} \quad (8)$$

The mereological predicates $\text{overlaps}$, $\text{proper part}$, $\text{sum}$, and $\text{difference}$ are defined in terms of $P$ in the standard ways.

$$DO_{xy} \equiv (\exists z)(P_{xz} \land P_{zy})$$
$$DP_{P} \equiv P_{xy} \land \neg P_{yx}$$
$$DS_{\text{sum}} \equiv (w)(O_{wz} \leftrightarrow (O_{wx} \lor O_{wy}))$$
$$D_{\text{Diff}} \equiv (w)(O_{wz} \leftrightarrow (\exists w_1)(P_{w_1x} \land \neg O_{w_1y} \land O_{w_1w}))$$

The usual axioms of reflexivity (A1), antisymmetry (A2), and transitivity (A3) are included as are the following usual existence axioms: if the region $x$ is not a part of region $y$ then there is a region $z$ such that $z$ is a difference of $y$ in $x$ (A4); for any two regions $x$ and $y$ there is a region $z$ that is the sum of $x$ and $y$ (A5). Axiom (A6) requires that the parthood predicate $P$ is crisp (Inwagen, 1981; Lewis, 1986; Sider, 2001; Hawley, 2002).

$$A1 \quad P_{xx} \quad A4 \quad \neg P_{xy} \rightarrow (\exists z)(\text{Diff}_{xyz})$$
$$A2 \quad P_{xy} \land P_{yx} \rightarrow x = y \quad A5 \quad (\exists z)(\text{Sum}_{xyz})$$
$$A3 \quad P_{xy} \land P_{yz} \rightarrow P_{xz} \quad A6 \quad D(P_{xy})$$

From these axioms it follows: the regions $x$ and $y$ are identical if and only if they overlap exactly the same regions; sums and differences are unique whenever they exist. Using axiom (A6) one can prove that the defined predicates $O$, $PP$, and $Diff$ are crisp.

EMR, an extensional mereology for regions with countable sums (Simons, 1987; Varzi, 1996), is the theory axiomatized by (A1-A6). It is well known that, axioms (A1-A5) are true in structures in which $P$ is interpreted as the subset relation among non-empty regular closed sets (Simons, 1987; Cohn et al., 1997). Thus axioms (A1-A5) are true in every $\Omega$-structure. Axiom (A6) is true in every $\Omega$-structure, since the interpretation of $P$ is the same at all precisification points as specified in Equation 8.

\(^3\)Clearly, (L1)-(L6) are rather general and apply to measures in various kinds of measure spaces. However, the models discussed here ($\Omega$-structures) are based on regular subsets of $\mathbb{R}^n$ and $\|d\|$ is understood as the volume of $d$. Since the volume of a regular subset of $\mathbb{R}^n$ is its Lebesgue measure, it seems justified to use the term ‘Lebesgue measure’ in the discussion of the models. The use of the term ‘Lebesgue measure’ rather than ‘volume’ allows to abstract from the dimensionality of the underlying space and, in addition, avoids confusion with notions such as ‘area’ or ‘length’.
4.3. Logical properties of crisp size predicates

EMR is extended by crisp qualitative size predicates. In the formal theory the same-size predicate \( \sim \) is introduced. At all precisification points the predicate \( \sim \) is interpreted as the relation of having the same Lebesgue measure:

\[
V(\sim) =_{df} \{ (d_1, d_2, \omega) \in D \times D \times \Omega \mid \|d_1\| = \|d_2\| \} \tag{9}
\]

In terms of \( P \) and \( \sim \) one can define that the size of \( x \) is less than or equal to the size of \( y \) if and only if there is a region \( z \) that is a part of \( y \) and has the same size as \( x \) (\( D \leq \)). The size predicate less than is defined in the standard way (\( D < \)).

\[
D \leq x \leq y \equiv (\exists z)(z \sim x \land P zy) \quad D < x < y \equiv x \leq y \land \neg(y \leq x)
\]

On the intended interpretation, \( x \leq y \) holds if and only if \( \|x\| \) is less than or equal to \( \|y\| \) and \( x < y \) holds if and only if \( \|x\| \) is strictly less than \( \|y\| \).

The following axioms are added:

\begin{align*}
A7 & \quad x \sim x \quad & A10 & \quad P xy \land x \sim y \rightarrow P yx \\
A8 & \quad x \sim y \rightarrow y \sim x \quad & A11 & \quad x \leq y \lor y \leq x \\
A9 & \quad x \sim y \land y \sim z \rightarrow x \sim z \quad & A12 & \quad x \leq y \land y \leq x \rightarrow x \sim y \\
& \quad D(x \sim y) \quad & A13 & \quad \text{true in virtue of (L5 on page 11)}
\end{align*}

Axioms (A7-A9) make \( \sim \) an equivalence relation. (A10) distinguishes \( \sim \) from the ‘same-diameter’ predicate. For example, a circle may have an ellipse with the same diameter as a proper part.\(^4\) Axioms (A11 and A12) make \( \leq \) a total pre-order. Axiom (A13) requires that \( \sim \) is crisp.\(^5\)

One can prove that \( \leq \) is reflexive and transitive (T1), that \( < \) is irreflexive, asymmetric, and transitive, and that \( \sim, \leq, \) and \( < \) are logically interrelated in the expected ways (T2-T4). In addition, one can prove that the defined size predicates \( \leq \) and \( < \) are crisp.

\begin{align*}
T1 & \quad x \leq y \land y \leq z \rightarrow x \leq z \\
T2 & \quad x \sim y \leftrightarrow x \leq y \land y \leq x \\
T3 & \quad x \leq y \land y \sim z \rightarrow x \leq z \\
T4 & \quad z \sim x \land x \leq y \rightarrow z \leq y
\end{align*}

The theory that extends EMR by the primitive \( \sim \) as well as \( D \leq, D < \), and (A7-A13) is called QSM. If \( \sim \) is interpreted as the relation of having the same Lebesgue measures as specified in Equation 9, then the above axioms are true in all \( \Omega \)-structures: All members of \( D \) (which is closed under finite sums and regularized complements) have a finite Lebesgue measure (L1-L3, L6 on page 11, see (Burk, 1997) for details). Since the domain of interpretation ranges over non-empty regular-closed sets, the Lebesgue measures of all regions of this domain are strictly positive (L4 on page 11). Axioms (A7-10) are true in virtue of the reflexivity, symmetry, and transitivity of the identity relation among real numbers. Axioms (A11-13) are true in virtue of the total order on the real numbers. Axiom (A10) is true in virtue of (L5 on page 11). (A13) is true, since the interpretation of \( \sim \) is the same at all precisification points.

\(^4\)Thus the axioms for "same size" are stronger than the axioms given in (Schmidtke and Woo, 2007) for "same size", where \( \sim \) was interpreted as the same-diameter relation.

\(^5\)Axioms (A7) - (A13) allow for the interpretation of \( \sim \) in a wide range of measure spaces. That is, they do not (and are not intended to) fix the interpretation of \( \sim \) to the identity of volumes (Lebesgue measures) of regular subsets of \( \mathbb{R}^n \).
(L1) All open and closed subsets of the Euclidean space \( \mathbb{R}^n \) have a Lebesgue measure.
(L2) If \( A \) is Lebesgue measurable, then so is its complement.
(L3) \( \| A \| \geq 0 \) for every Lebesgue measurable set \( A \).
(L4) The Lebesgue measure is strictly positive on non-empty open sets. (All the subsets of \( \mathbb{R}^n \) whose dimension is smaller than \( n \) have null Lebesgue measure in \( \mathbb{R}^n \). For instance, straight lines or circles have null Lebesgue measure in \( \mathbb{R}^2 \).)
(L5) If \( A \) and \( B \) are Lebesgue measurable and \( A \) is a subset of \( B \), then \( \| A \| \leq \| B \| \).
(L6) Countable unions and intersections of Lebesgue measurable sets are Lebesgue measurable.

5. Vague size predicates and their semantics

The theory QSM is extended by the primitive relational predicate \( \approx \), which serves as symbolic representation of the vague predicate "has roughly the same size as". The resulting theory is called VSM. In terms of \( \approx \) other size predicates are defined and their semantic properties are analyzed. This includes semantic concepts such as Sorites series and Sorites-style paradox. The semantic analysis of this section will provide the motivation for the axiomatization in the later parts of the paper and will also establish links to related approaches in the literature.

5.1. The predicate "has roughly the same size as"

The interpretation function \( V \) maps the predicate \( \approx \) onto the following subset of \( D \times D \times \Omega \):

\[
V(\approx) = \{ (d_1, d_2, \omega) \in D \times D \times \Omega \mid 1/(1 + \omega) \leq \|d_1\|/\|d_2\| \leq 1 + \omega \} 
\]  

(10)

Consider Figure 1, which depicts the interpretations of \( \sim, \approx, \ll, \) and \( \prec \) for \( \omega = 0.3 \) (left) and \( \omega = 0.1 \) (right). If the regions \( d_1 \) and \( d_2 \) have exactly the same size, then – according to Equation 9 – \( \|d_1\|/\|d_2\| \) represents a point lying on a line (not depicted) in the centre of the corridor delimited by the dotted and the solid lines. If \( d_1 \) and \( d_2 \) have roughly the same size, then – according to Equation 10 – \( \|d_1\|/\|d_2\| \) represents a point lying within the area delimited by the dotted and the solid lines. The parameter \( \omega \) determines the precisified interpretation of \( \approx \) by specifying the area delimited by the dotted and the solid lines. The more \( \omega \) approaches 0 the more narrow the delimited area becomes.

Fig. 1. Interpretations of \( \sim, \approx, \ll, \) and \( \prec \) for \( \omega = 0.3 \) (left) and \( \omega = 0.1 \) (right). (Both diagrams use a logarithmic scale.)

Let \( \omega_\downarrow \) and \( \omega_\uparrow \) respectively be the greatest lower bound and the least upper bound of \( \Omega \). From the interpretation of \( \approx \) in Equation 10 it follows that the tuple \( \langle d_1, d_2, \omega_\downarrow \rangle \) is a member of \( V(\approx) \) if and only
if for all \( \omega \in \Omega: \langle d_1, d_2, \omega \rangle \) is a member of \( V(\approx) \). Similarly, the tuple \( \langle d_1, d_2, \omega_1 \rangle \) is not a member of \( V(\approx) \) if and only if for no \( \omega \in \Omega: \langle d_1, d_2, \omega \rangle \) is a member of \( V(\approx) \). That is:

\[
\begin{align*}
\langle d_1, d_2, \omega_1 \rangle &\in V(\approx) \text{ iff } \forall \omega \in \Omega: \langle d_1, d_2, \omega \rangle \in V(\approx) \\
\langle d_1, d_2, \omega_1 \rangle &\not\in V(\approx) \text{ iff } \forall \omega \in \Omega: \langle d_1, d_2, \omega \rangle \not\in V(\approx)
\end{align*}
\]

From the definition of truth conditions for formulas of the forms \( U \alpha \) and \( S \alpha \) in Equation 5 it then follows:

**Lemma 1.** \( V(U(x \approx y), \omega) = 1 \) iff \( V(x \approx y, \omega_1) = 1 \). \( V(U(x \approx y), \omega) = 0 \) iff \( V(x \approx y, \omega_1) = 0 \). \( V(S(x \approx y), \omega) = 1 \) iff \( V(x \approx y, \omega_1) = 1 \). \( V(S(x \approx y), \omega) = 0 \) iff \( V(x \approx y, \omega_1) = 0 \).

It also follows that the positive extension of the binary predicate \( \approx \) is the set of all tuples \( \langle d_1, d_2, \omega_1 \rangle \) such that \( \langle d_1, d_2, \omega_1 \rangle \) is a member of \( V(\approx) \). The negative extension of \( \approx \) is the set of all tuples \( \langle d_1, d_2, \omega_1 \rangle \) such that \( \langle d_1, d_2, \omega_1 \rangle \) is not a member of \( V(\approx) \).

\[
\delta(\approx)^+ = \{ \langle d_1, \ldots, d_n \rangle \in D_1 \times \cdots \times D_n \mid \langle d_1, \ldots, d_n, \omega_1 \rangle \in V(\approx) \}
\]

\[
\delta(\approx)^- = \{ \langle d_1, \ldots, d_n \rangle \in D_1 \times \cdots \times D_n \mid \langle d_1, \ldots, d_n, \omega_1 \rangle \not\in V(\approx) \}
\]

The positive and negative extensions of \( \approx \) for \( \Omega = [\frac{1}{3}, \frac{1}{2}] \) (left) and \( \Omega = [\frac{1}{3}, \frac{1}{2}] \) (right) are depicted in Figure 2.

![Fig. 2. The positive extension (\( \delta(\approx)^+ \)) and the negative extension (\( \delta(\approx)^- \)) of \( \approx \) for \( \Omega = [\frac{1}{3}, \frac{1}{2}] \) (left) and \( \Omega = [\frac{1}{3}, \frac{1}{2}] \) (right).](image)

### 5.2. Defined vague size predicates

In terms of the primitive \( \approx \) ("has roughly the same size as") one can define: the region \( x \) is **negligible in size with respect to** region \( y \) if and only if there are regions \( z_1 \) and \( z_2 \) such that (i) \( x \) and \( z_1 \) have the same size, (ii) \( z_1 \) is a part of \( y \), (iii) \( z_2 \) is the difference of \( z_1 \) in \( y \) and (iv) \( z_2 \) and \( y \) have roughly the same size (\( D \ll \)). The predicates **less than or roughly equal in size to** (\( \leq \)) and **less than and not roughly equal in size to** (\( \prec \)) are defined in the standard ways (\( D \leq \) and \( D \prec \)).

\[
\begin{align*}
D \ll & \quad x \ll y \equiv (\exists z_1)(\exists z_2)(z_1 \sim x \land P z_1 y \land Diff y z_1 z_2 \land z_2 \approx y) \\
D \leq & \quad x \leq y \equiv x \leq y \vee x \approx y \\
D \prec & \quad x \prec y \equiv x \leq y \land \neg(y \leq x)
\end{align*}
\]

Consider, again, Figure 1 which also depicts interpretations of the predicates \( \ll, \leq, \) and \( \prec \) for \( \omega = 0.3 \) (left) and \( \omega = 0.1 \) (right). If \( d_1 \) is negligible in size with respect to \( d_2 \), then \( ||d_1||, ||d_2|| \) represents a point lying above the dashed line. The constraints that determine the interpretations of the defined predicates in \( \Omega \)-structures are listed in Equation 13.
V(\(\preceq\)) = \{ (d_1, d_2, \omega) \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \|d_1\|/\|d_2\| < \omega/(1 + \omega) \}
V(\(\preceq\)) = \{ (d_1, d_2, \omega) \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \|d_1\|/\|d_2\| \leq 1 + \omega \}
V(\(\prec\)) = \{ (d_1, d_2, \omega) \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \|d_1\|/\|d_2\| < 1/(1 + \omega) \} \tag{13}

If \(\Omega = \{0\}\) then \(V(\sim) = V(\approx)\) and \(V(\preceq) = \emptyset\). If \(\Omega = \{1\}\) then \(V(\prec) = V(\preceq)\). Those special cases are excluded by restricting \(\Omega\) to non-empty subsets of the open interval \((0, 1)\), i.e., \(\emptyset \subset \Omega \subset (0, 1)\). On all precisation points within \((0, 1)\), \(V(\approx)\) is a proper, non-empty sub-relation of \(V(\prec)\). Additional restrictions on \(\Omega\) that are imposed by the axioms introduced in Sections 5 & 6 will be discussed in Section 7.

From the interpretations of \(\preceq\) and \(\prec\) in Equation 13 it follows:
\[
\begin{align*}
\langle d_1, d_2, \omega_1 \rangle \in V(\preceq) & \iff \forall \omega \in \Omega : \langle d_1, d_2, \omega \rangle \in V(\preceq) \\
\langle d_1, d_2, \omega_1 \rangle \in V(\prec) & \iff \forall \omega \in \Omega : \langle d_1, d_2, \omega \rangle \in V(\prec)
\end{align*}
\tag{14}
\]
That is, the tuple \(\langle d_1, d_2, \omega_1 \rangle\) is a member of \(V(\preceq)\) if and only if for all \(\omega \in \Omega\), \(\langle d_1, d_2, \omega \rangle\) is a member of \(V(\preceq)\). By contrast, the tuple \(\langle d_1, d_2, \omega_1 \rangle\) is a member of \(V(\prec)\) if and only if for all \(\omega \in \Omega\), \(\langle d_1, d_2, \omega \rangle\) is a member of \(V(\prec)\). From the truth conditions in Equation 5 it then follows:

**Lemma 2.** \(V(U(x \ll y), \omega) = 1\) iff \(V(x \ll y, \omega) = 1\); \(V(U(x \ll y), \omega) = 0\) iff \(V(x \ll y, \omega) = 0\);
\(V(S(x \ll y), \omega) = 1\) iff \(V(x \ll y, \omega) = 1\); \(V(S(x \ll y), \omega) = 0\) iff \(V(x \ll y, \omega) = 0\);
\(V(U(x \ll y), \omega) = 1\) iff \(V(x \ll y, \omega) = 1\). \(V(U(x \ll y), \omega) = 0\) iff \(V(x \ll y, \omega) = 0\).
\(V(S(x \ll y), \omega) = 1\) iff \(V(x \ll y, \omega) = 1\). \(V(S(x \ll y), \omega) = 0\) iff \(V(x \ll y, \omega) = 0\).

The positive and the negative extensions of the predicates \(\preceq\) and \(\prec\) can be defined in terms of \(\omega_1\) and \(\omega\).

### 5.3. Sorites series

Let \(F\) be a meta-variable that ranges over vague unary predicates in the object language of VSM and let \(\delta(F)^{+}\) and \(\delta(F)^{-}\) respectively be the positive the negative extensions of \(F\) as defined in Eqn. 6. In addition, let the meta-variable \(\vdash \subseteq \mathcal{D} \times \mathcal{D}\) range over relations of ‘irrelevant difference’ (e.g., (Williamson, 1988; Pinkal, 1995; Hyde, 1996)) with respect to the vague predicate \(F\). In the literature, \(F\) often stands for unary predicates such as near, far, red, or tall and \(\vdash\) often stands for binary relations such as roughly-as-distant-as, is-indistinguishable-in-color-with-respect-to, or roughly-the-same-height-as (e.g., (Pinkal, 1995; Keefe and Smith, 1996; Fara, 2000)). A \(F\)-Sorites sequence can be defined as follows:

**Definition 2.** A sequence \(d_1, d_2, \ldots, d_i, \ldots, d_j, \ldots, d^*\) of domain members is a \(F\)-Sorites sequence if and only if: (a) immediately neighboring members \(d_i, d_j\) of the sequence \(d^+, \ldots, d_i, d_j, \ldots, d^*\) are related by the relation of ‘irrelevant difference’ \(\vdash\); (b) \(d^+ \in \delta(F)^{+}\) is a member of the positive extension of \(F\), and (c) \(d^- \in \delta(F)^{-}\) is a member of the negative extension of \(F\) (or vice versa) (Barnes, 1982; Keefe and Smith, 1996; Hyde, 1996). For two domain members \(d^+\) and \(d^-\) there may potentially be infinitely many finite sequences of domain members that form an \(F\)-Sorites sequence from \(d^+\) to \(d^-\). A \(F\)-Sorites series from \(d^+\) to \(d^-\) is the class of finite \(F\)-Sorites sequences from \(d^+\) to \(d^-\).

In this paper ‘\(F\)-Sorites series’ stands for the more restricted class of Sorites series in \(\Omega\)-structures where the vague predicate \(F\) is definable in terms of the size predicates \(\approx\) and \(\preceq\) using the following types of lambda expressions:

\[\begin{align*}
D_{F^+} &\quad F^+ \equiv \lambda x. \ (d^+ \approx x) \\
D_{F^-} &\quad F^+ \equiv \lambda x. \ (x \ll d^-) \\
D_{F^*} &\quad F^* \equiv \lambda x. \ (d^+ \ll x)
\end{align*}\]

\(^6\)Other cases such as \(F^* \equiv \lambda x. \ (x \ll d^-)\) and \(F^{**} \equiv \lambda x. \ (d^+ \ll x)\) are omitted.
For example, $F'$ is an abbreviation for the unary predicate "has roughly the same size as $d^+$" where $d^+$ is the first element of a $F'$-Sorites series.

$F$-Sorites sequences of this kind are required to have the following additional properties:

1. **Crissipness of 'irrelevant difference'**: The relation of 'irrelevant difference' is identical to the positive extension of the vague predicate $\approx$, i.e., $\bowtie \subseteq \delta(\approx)^+$. That is, despite the vagueness of the predicate $\approx$, whether or not immediately neighboring members of a $F$-Sorites sequence are in the relation of 'irrelevant difference' ($\approx$) is not subject to vagueness.

2. **'Non-significant difference' property**: Close but not immediately neighboring members $d_i, d_k$ of a $F$-Sorites sequence $d^+, \ldots, d_i, d_j, d_k, \ldots, d^-$ are related in a weaker sense: for such sequence members the vague predicate $\approx$ holds at least on some precisification, i.e., $(d_i, d_k) \notin \delta(\approx)^-$. Since $\bowtie \subseteq \delta(\approx)^+, d^+ \in \delta(\approx)^+$ and $d^- \in \delta(\approx)^-$, it follows that the predicate $\approx$ is not transitive. Since there are domain members $d_i, d_k$ in $F$-Sorites sequence $d^+, \ldots, d_i, d_j, d_k, \ldots, d^-$ such that $(d_i, d_k) \notin \delta(\approx)^-$, it follows that $\approx$ is not intransitive. It therefore follows that the vague predicate $\approx$, whose positive extension serves as the relation of 'irrelevant difference' in $F$-Sorites series in $\Omega$-structures, can be neither transitive nor intransitive but has to be **weakly transitive** in the following sense:

**Definition 3.** A binary predicate $R$ is weakly transitive in $\mathcal{D}$ if and only if for all $x, y, z \in \mathcal{D}$: if $(x, y) \in \delta(R)^+$ and $(y, z) \in \delta(R)^+$, then $(x, z) \notin \delta(R)^-$. Obviously, if $R$ is crisp, then $R$ is transitive if and only if $R$ is weakly transitive. Hence, if the relation of 'irrelevant difference' in a $F$-Sorites series is formed by the extension of a crisp predicate then this crisp predicate is intransitive. Examples include predicates such as "has one hair less than", "has one grain less/more than", etc.

### 5.4. Sorites-style paradoxes

In addition to the logical properties of the predicates that are associated with Sorites series, one also has to address apparently sound inferences that seem to be justified by intuitions which seem to be essential to the understanding of vague predicates and associated Sorites series.

Consider $F$-Sorites series between domain members $d_1$ and $d_n$ with the properties specified in the previous section. The form of an argument whose instances, if sound, lead to Sorites-style *paradoxes* is depicted in Equation 15. Arguments of this form explicate inferences that are based on semantic intuitions and are, therefore, stated in the meta-language of VSM. (For a more general treatment see, for example, (Pinkal, 1995; Hyde, 1996).)

\[
\begin{align*}
(s_0) & \quad \text{There is a } F\text{-Sorites sequence from } d_1 \text{ to } d_n \\
(s_1) & \quad d_1 \in \delta(F)^+ \quad \text{iff} \quad \mathsf{V}(F(d_1)) = 1 \\
(s_2) & \quad \text{if } d_i \bowtie d_2 \quad \text{then} \quad \mathsf{V}(F(d_1 \rightarrow F(d_2))) = 1 \\
& \quad \text{if } \ldots \\
(s_{n-1}) & \quad d_{n-1} \bowtie d_n \quad \text{then} \quad \mathsf{V}(F(d_{n-1} \rightarrow F(d_n))) = 1 \\
(s_n) & \quad d_n \in \delta(F)^- \quad \text{iff} \quad \mathsf{V}(\neg F(d_n)) = 1 \\
(s_{n+1}) & \quad \text{if } \mathsf{V}(F(d_1 \rightarrow F(d_2))) = 1 \quad \text{and} \quad \ldots \quad \text{and} \quad \mathsf{V}(F(d_{n-1} \rightarrow F(d_n))) = 1 \quad \text{then} \\
& \quad \mathsf{V}(F(d_1 \rightarrow F(d_n))) = 1 \\
(c) & \quad \mathsf{V}(F\neg d_n \land \neg F\neg d_n) = 1
\end{align*}
\]

Clearly, the form of the argument that is depicted in Equation 15 is valid with respect to the underlying semantics in the sense that if all premises $(s_0) - (s_{n+1})$ are true, then the conclusion $(c)$ cannot be false. Clearly, there are $F$-Sorites series and thus the premise $(s_0)$ is true. In addition, it is clear that the premises $(s_1), (s_n),$ and $(s_{n+1})$ are all true for any $F$-Sorites series from $d_1$ to $d_n$. However, for every $F$-Sorites series $d_1, \ldots, d_n$ one can show that premisses of the forms $(s_2) \ldots (s_{n-1})$ are not all true for vague predicates of the form $F', F''$, and $F'''$:
Lemma 3. For every $F$-Sorites series $d^+ \ldots d^-$ there are domain members $d_k, d_l$ such that:

$$V(U(d_k \approx d_l)) = 1$$

and

$$V(U(F d_k)) = 1 \quad \text{and} \quad V(U(F d_l)) = 0.$$ 

Proof. $(F = F')$: Show that there are domain members $d_k$ and $d_l$ such that

$$V(U(d_k \approx d_l)) = 1 \quad \text{and} \quad V(U(d^+ \approx d_k)) = 1 \quad \text{and} \quad V(U(d^+ \approx d_l)) = 0. \quad (16)$$

In $\Omega$-structures, for any given domain member $d^+$, there are domain members $d_k, d_l$, such that either

(i) $1/(1 + \omega_1) = \|d^+\|/\|d_k\|$ and $1/(1 + \omega_1) \leq \|d_k\|/\|d_l\| < 1$ or (ii) $\|d^+\|/\|d_k\| = (1 + \omega_1)$ and $1 < \|d_k\|/\|d_l\| \leq 1 + \omega_1$. Algebraic computations show that for (i) $1/(1 + \omega_1)^2 \leq \|d^+\|/\|d_l\| < 1/(1 + \omega_1)$ and for (ii) $1 + \omega_1 < \|d^+\|/\|d_l\| \leq (1 + \omega_1)^2$. That is, in both cases, (i) and (ii), it is not the case that $1/(1 + \omega_1) \leq \|d^+\|/\|d_l\| \leq (1 + \omega_1)$. By Def. 10, $V(d_k \approx d_l, \omega_1) = 1$, $V(d^+ \approx d_k, \omega_1) = 1$ and $V(d^+ \approx d_l, \omega_l) = 0$. By Lemma 1, $V(U(d_k \approx d_l, \omega)) = 1$, $V(U(d^+ \approx d_k, \omega)) = 1$ and $V(U(d^+ \approx d_l, \omega)) = 0$ for any precisification $\omega \in \Omega$. Hence, $V(U(d_k \approx d_l)) = 1$, $V(U(d^+ \approx d_k)) = 1$ and $V(U(d^+ \approx d_l)) = 0$.

$(F = F'')$: Show that there are domain members $d_k, d_l$ such that

$$V(U(d_k \approx d_l)) = 1 \quad \text{and} \quad V(U(d_k \ll d^+)) = 1 \quad \text{and} \quad V(U(d_l \ll d^-)) = 0. \quad (17)$$

In $\Omega$-structures, for any given domain member $d^-$ there are domain members $d_k$ and $d_l$ that satisfy the following constraints: $1/(1 + \omega_1) \leq \|d_1\|/\|d_l\| < 1$, $\|d_1\|/\|d_l\| < \omega_1/(1 + \omega_1)$ and $\|d_l\|/\|d^-\| = \omega_1/(1 + \omega_1)$. By Defs. 10 and 13: $V(d_k \approx d_l, \omega_1) = 1$, $V(d^+ \approx d_l, \omega_1) = 1$ and $V(d_l \ll d^-, \omega_1) = 0$. By Lemmata 1 and 2, $V(U(d_k \approx d_l, \omega)) = 1$, $V(U(d_k \ll d^-), \omega) = 1$ and $V(U(d_l \ll d^-), \omega) = 0$ for any $\omega \in \Omega$. That is, $V(U(d_k \approx d_l)) = 1$, $V(U(d_k \ll d^-)) = 1$ and $V(U(d_l \ll d^-)) = 0$.

$(F = F'')$: Show that there are domain members $d_k, d_l$ such that

$$V(U(d_k \approx d_l)) = 1 \quad \text{and} \quad V(U(d^+ \ll d_k)) = 1 \quad \text{and} \quad V(U(d^+ \ll d_l)) = 0. \quad (18)$$

In $\Omega$-structures, for any given domain member $d^+$ there are domain members $d_k$ and $d_l$ that satisfy the following constraints: $1 < \|d_1\|/\|d_l\| \leq 1 + \omega_1$, $\|d^+\|/\|d_k\| < \omega_1/(1 + \omega_1)$ and $\|d^+\|/\|d_l\| = \omega_1/(1 + \omega_1)$. By Defs. 10 and 13, $V(d_k \approx d_l, \omega_1) = 1$, $V(d^+ \approx d_l, \omega_1) = 1$ and $V(d_l \ll d^-, \omega_1) = 0$. By Lemmata 1 and 2, $V(U(d_k \approx d_l, \omega)) = 1$, $V(U(d^+ \ll d_k), \omega) = 1$ and $V(U(d^+ \ll d_l), \omega) = 0$ for any $\omega \in \Omega$. Thus, $V(U(d_k \approx d_l)) = 1$, $V(U(d^+ \ll d_k)) = 1$ and $V(U(d^+ \ll d_l)) = 0$. ∎

This shows that in arguments of the form depicted in Equation 15 not all of the premisses $(s_2) \ldots (s_{n-1})$ can be true. Therefore, arguments of this form are unsound for every $F$-Sorites sequence $d^+, \ldots, d^-$. Note, that the proof of Lemma 3 indicates that there is a sharp cut-off point which seems to conflict with the usual conception of vagueness as a semantic phenomenon in which the existence of such cut-off point is denied (e.g., (Hyde, 1996)). This sharp cut-off point is a consequence of the crispness of the positive and negative extensions of the predicate $\approx$ in $\Omega$-structures. As pointed out above, the shortcoming of crisp positive and negative extensions is addressed in the extended framework presented in (Bittner, 2011a).

6. Logical properties of vague size predicates

As a starting point axioms that specify some uncontroversial axioms for vague size predicates are included in the formal theory. Then, vagueness-specific properties of the predicates $\approx$, $\geq$, $\ll$, and $\prec$ are formalized.
6.1. Basic properties

The predicate \( \approx \) is reflexive (A14) and symmetric (A15).

\[
\begin{align*}
A14 & \quad x \approx x \\
A15 & \quad x \approx y \rightarrow y \approx x
\end{align*}
\]

One can prove that \( \preceq \) is reflexive, that \( \prec \) and \( \ll \) are asymmetric and irreflexive, and that \( \prec \) and \( \ll \) both imply \( \prec \) (T5, T6), that \( \sim \) implies \( \approx \) and that \( \preceq \) implies \( \leq \). However, it is not the case that \( \approx \) implies \( \sim \), nor that \( \preceq \) implies \( \leq \), nor that \( \prec \) implies \( \prec \). This can be verified by inspecting Figure 1. One can also prove that, to a certain degree, the vague predicates \( \preceq \), \( \approx \), and \( \prec \) mirror the logical properties of their crisp counterparts \( \leq \), \( \sim \), and \( \prec \) (T7-T9).

\[
\begin{align*}
T5 & \quad x < y \rightarrow x < y \\
T6 & \quad x \ll y \rightarrow x < y \\
T7 & \quad x \preceq y \lor y \preceq x
\end{align*}
\]

The following domain members are required to exist: for every \( x \) there is a \( y \) such that unequivocally, \( x \) is negligible with respect to \( y \) (A16); for every \( x \) there is a \( y \) such that unequivocally, \( x \) and \( y \) have roughly the same but distinct sizes (A17).

\[
\begin{align*}
A16 & \quad (\exists y)(U(x \ll y)) \\
A17 & \quad (\exists y)(U(x \approx y) \land \neg(x \sim y))
\end{align*}
\]

Thus, there is no finite upper bound on sizes and the positive extension of \( \sim \) is a proper sub-relation of the positive extension of \( \approx \), i.e., \( \delta(\sim)^\uparrow \subset \delta(\approx)^\uparrow \). A sentence of the form \((\exists y)(U(y \ll x))\) is not included as an axiom, since this sentence is not true in models where there are smallest units or greatest lower bounds.

6.2. Logical interrelations between crisp and vague size predicates

The axioms (A18) and (A19) characterize the interrelations of the vague predicates \( \approx \) and \( \ll \) with the crisp predicate \( \leq \):

\[
\begin{align*}
A18 & \quad x \approx y \land x \leq z \land z \leq y \rightarrow (z \approx x \land z \approx y) \\
A19 & \quad x \ll y \land y \leq z \rightarrow x \ll z
\end{align*}
\]

Using (A18) and (A19) one can prove the following theorems about various forms of the logical composition of the crisp and vague size predicates \( \sim \), \( \approx \), \( \leq \), and \( \ll \) (T10-T14).

\[
\begin{align*}
T10 & \quad x \sim y \land y \approx z \rightarrow x \approx z \\
T11 & \quad x \approx y \land y \sim z \rightarrow x \approx z \\
T12 & \quad x \leq y \land y \ll z \rightarrow x \ll z \\
T13 & \quad x \leq y \land y \leq z \rightarrow x \leq z \\
T14 & \quad x \ll y \land y \leq z \rightarrow x \leq z
\end{align*}
\]

In general, logical compositions of binary relational predicates have the form: \( R \ xy \land S \ yz \rightarrow T \ xz \). Important special cases include \( R = T \), \( S = T \), and \( R = S = T \) (transitivity).

Theorems (T10-T14) illustrate that in the development of the formal theory important logical properties of vague size predicates are explicated in relationship to crisp size predicates via their logical composition. That is, crisp predicates provide a logical basis with respect to which the logical properties of vague predicates are specified via compositional sentences.
6.3. Boundary cases

Two axioms are included in the formal theory requiring that for all \( x \) there is a \( y \) such that the ordered pair \( \langle x, y \rangle \) is a boundary case for \( \preccurlyeq \) in the sense of Definition 1 (A20). Similarly for \( \prec \) (A21). One can then prove that the predicates \( \approx \) and \( \preceq \) are both vague (T15–T16).

\[
\begin{align*}
A20 & \land (\exists y)\neg(\preceq y) & T15 & \land (\exists x)(\exists y)\lnot(\approx x y) \\
A21 & \land (\exists y)\neg(\prec y) & T16 & \land (\exists x)(\exists y)\lnot(\preceq x y)
\end{align*}
\]

Let the ordered pair \( \langle d_1, d_2 \rangle \) be a boundary case for the binary predicate \( \approx \). Clearly, if \( \langle d_1, d_2 \rangle \) is a boundary case for \( \approx \) then \( \langle d_1, d_2 \rangle \) is not a boundary case for \( \preceq \). That is, if \( \lnot(\preceq d_1 d_2) \) then not \( \lnot(\approx d_1 d_2) \). Unfortunately, this is not provable yet in the presented formal theory. For this reason an axiom is included requiring that if \( x \) is negligible in size with respect to \( y \) on some precisification, then unequivocally, \( x \) is smaller than \( y \) and \( x \) and \( y \) are not of roughly the same size (A22). It immediately follows that if \( x \) negligible in size with respect to \( y \) on some precisification then unequivocally, \( x \) and \( y \) are not of roughly the same size (T17).

\[
\begin{align*}
A22 & \land S(x \preceq y) \rightarrow U(x \prec y) & T17 & \land S(x \preceq y) \rightarrow U(\neg x \approx y)
\end{align*}
\]

One can then prove: if the ordered pair \( \langle d_1, d_2 \rangle \) is a boundary case for \( \approx \) then \( \langle d_1, d_2 \rangle \) is not a boundary case for \( \preceq \) (T18); \( \langle d_1, d_2 \rangle \) is a boundary case for the predicate \( \approx \) if and only if \( \langle d_1, d_2 \rangle \) is a boundary case for \( \prec \) (T19).

\[
\begin{align*}
T18 & \land l(x \preceq y) \rightarrow \lnot l(x \approx y) & T19 & \land l(x \approx y) \leftrightarrow (l(x \prec y) \lor l(y \prec x))
\end{align*}
\]

6.4. "Has roughly the same size as" as ‘irrelevant difference’

Consider potential Sorites-style paradoxes resulting from arguments of the form displayed in Equation 15. In addition, let the vague predicate \( F = F' \) be an abbreviation for the unary predicate “has roughly the same size as \( d^+ \)” \( (D_F) \), and let the relation of ‘irrelevant difference’ for \( F' \) be the positive extension of the vague predicate \( \approx \), i.e., \( \approx = \delta(\approx) \). As pointed out in Lemma 3(16), on the proposed semantics \( F' \)-Sorites-style paradoxes do not occur because not all premises of the argument are true. To make this explicit in the formal theory, Axiom (A23) is included:

\[
A23 \land U(\approx x y) \land \lnot(x \sim y) \land (\exists z)(U(\approx x z) \land \lnot U(\approx z y))
\]

For every vague predicate of the form \( F' \) one then can prove a particular instance of the following schema:

\[
TS20 \land U(F' x) \rightarrow (\exists y)(U(\approx x y) \land \lnot U(F' y)),
\]

where \( F' \equiv \lambda x. (d^+ \approx x \land \lnot(d^+ \sim x)) \). Thus, for any \( F' \)-Sorites series and any given choice of \( F' \) there is a theorem in VSM that shows that not all the premisses of the corresponding argument of the form displayed in Equation 15 can be true.

One can prove that \( \approx \) is not transitive (T21).

\[
T21 \land (\exists x)(\exists y)(\exists z)(U(\approx x y) \land U(\approx y z) \land \lnot U(\approx x z))
\]

To insure that \( \approx \) is weakly transitive in the sense of Def. 3, Axiom (A24) is included in the formal theory.

\[
A24 \land U(\approx x y) \land U(\approx y z) \rightarrow S(\approx x z)
\]

One can then prove theorems (T22) and (T23):
\[ T22 \quad \mathbf{U}(x \approx y) \rightarrow \mathbf{S}(x \approx z \rightarrow y \approx z) \]
\[ T23 \quad \mathbf{U}(x \approx y) \rightarrow \mathbf{S}(F' x \rightarrow F' y) \]

Clearly, the inferences warranted by theorems (T22) and (T23) do not give rise to \( F' \)-Sorites-style paradoxes. Consequently, axioms (A23) and (A24) jointly constrain the interpretations of the vague predicate \( \approx \) in a way that captures the underlying semantic intuitions.

### 6.5. Logical composition of vague size predicates

Consider the argument form underlying potential \( F \)-Sorites-style paradoxes in Equation 15, let \( F = F'' \) (respectively \( F = F''' \)), and let \( \vartriangleleft = \delta(\approx)^{+} \). As pointed out in Lemma 3(17),(18), on the proposed semantics \( F''-(F''' \)-Sorites-style paradoxes do not occur. To make this explicit, the axioms (A25) and (A26) are included in the formal theory: (Axiom (A26) makes the axiom (A16) redundant.)

\[ A25 \quad (\exists x)(\mathbf{U}(x \prec z) \rightarrow (\exists x)(\exists y)(\mathbf{U}(x \approx y) \land \mathbf{U}(x \prec z) \land \neg \mathbf{U}(y \prec z)) \]
\[ A26 \quad (\exists x)(\exists y)(\mathbf{U}(x \approx y) \land \mathbf{U}(z \prec x) \land \neg \mathbf{U}(z \prec y)) \]

It immediately follows, that for any given choice of \( F'' \) and \( F''' \) there is a theorem in the formal theory that ensures that not all the premisses of the argument in Equation 15 can be true.

\[ T_{S24} \quad (\exists x)(\mathbf{U}(x \prec z) \rightarrow (\exists x)(\exists y)(\mathbf{U}(x \approx y) \land \mathbf{U}(F' x) \land \neg \mathbf{U}(F' y))) \]
\[ T_{S25} \quad (\exists x)(\exists y)(\mathbf{U}(x \approx y) \land \mathbf{U}(F'' x) \land \neg \mathbf{U}(F'' y)) \]

Clearly, compositional sentences of the form

\[ * \quad x \prec y \land y \approx z \rightarrow x \prec z \]
\[ ** \quad x \approx y \land y \approx z \rightarrow x \prec z \]

are inconsistent with axioms (A25) and (A26) and, therefore, cannot be theorems of VSM. However, the following weakened versions of (\*) and (**\*) are included as axioms (A27–A30).

\[ A27 \quad \mathbf{U}(x \prec y) \land \mathbf{U}(y \approx z) \rightarrow \mathbf{S}(x \prec z) \]
\[ A28 \quad \mathbf{U}(x \approx y) \land \mathbf{U}(y \approx z) \rightarrow \mathbf{S}(x \prec z) \]
\[ A29 \quad x \prec y \land y \approx z \rightarrow x \prec z \]
\[ A30 \quad x \approx y \land y \approx z \rightarrow x \prec z \]

Clearly, (A27) and (A28) are compositional counterparts of the weak transitivity axiom (A24). Jointly, Axioms (A22) and (A25–A30) constrain the logical interrelations of \( \prec, \approx \), and \( \prec \) in a way that captures important logical and semantical intuitions. This claim is supported by a number of consequences. Among others one can then prove the following theorems about the logical interrelations and the composition of \( \prec, \approx \), and \( \prec \) (T26–T33).

\[ T26 \quad x \prec y \rightarrow \neg x \approx y \]
\[ T27 \quad x \prec y \rightarrow x \prec y \]
\[ T28 \quad x \prec y \land y \approx z \rightarrow x \prec z \]
\[ T29 \quad x \prec y \land y \prec z \rightarrow x \prec z \]
\[ T30 \quad x \prec y \land y \prec z \rightarrow x \prec z \]
\[ T31 \quad x \prec y \land y \prec z \rightarrow x \prec z \]
\[ T32 \quad x \prec y \land y \prec z \rightarrow x \prec z \]
\[ T33 \quad x \prec y \land y \prec z \rightarrow x \prec z \]

Note that theorems (T28), (T29) demonstrate that there are vague size predicates such as \( \prec \) and \( \prec \) that are transitive. The predicates \( \prec \) and \( \prec \) are vague but since their positive extensions do not serve as a relation of ‘irrelevant difference’ in some Sorites series, their transitivity does not give rise to contradictions.

Note also, that there cannot be a stronger version of theorem (T32) of the form \( x \prec y \land y \approx z \rightarrow x \prec z \). To see this, assume that \( V(x \prec y \land y \approx z, \omega) = 1 \). By Definitions 9 and 10: \( ||x||/||y|| < 1/1 + \omega \) and \( 1/(1 + \omega) \leq ||y||/||z|| \leq 1 + \omega \). Let \( ||y||/||z|| = 1 + \omega \). Then \( ||x||/||z|| < 1 \). That is, there are \( x \) and \( z \) such that \( 1/(1 + \omega) \leq ||x||/||z|| \leq 1 \) such that \( V(x \prec y \land y \approx z, \omega) = 1 \) and \( V(x \prec z, \omega) = 0 \). Similarly for Theorem (T33).
Consider Table 2 which summarizes important compositional theorems of VSM. (This table is not intended to serve as a composition table for automated reasoning purposes in the sense of (Renz and Nebel, 1999) or (Eschenbach, 2001).) The top left quadrant of the interior the table contains theorems about the composition of crisp size predicates. These theorems belong to QSM, the crisp core of VSM (Section 4.3). The theorems in the top right and the bottom left quadrants are such that vague and crisp size predicates are composed. These theorems explicate logical properties of vague size predicates in relationship to crisp size predicates via their logical composition (Section 6.1).

The bottom right quadrant of Table 2 lists theorems about the composition of vague size predicates. There are empty slots in this quadrant indicating that corresponding compositional sentences cannot be axioms or theorems of VSM. This primarily affects compositional sentences that include the vague predicates $\approx$ and $\lessapprox$. The inclusion of any of those sentences would give rise to inconsistencies, because the positive extension of $\approx$ serves as relation of ‘irrelevant difference’ in $F$-Sorites series. The table also shows that vague binary predicates such as $\lessapprox$ and $\lesslessapprox$ can be composed without giving rise to contradictions. Consequently, it is important to distinguish vague predicates that serve as relation of ‘irrelevant difference’ in $F$-Sorites series from vague predicates that do not.

![Table 2](https://example.com/table2.png)

**Table 2**

Important compositional theorems of VSM.

### 7. Logically admissible precisifications

For the axioms of the theory VSM to be true in $\Omega$-structures, the precisification points in $\Omega$ need to be within logically admissible ranges. In this section it is shown that the axioms of VSM are true in $\Omega$-structures, $(\Omega, \mathcal{D}, \mathcal{V}, \subseteq, \|\|)$, where $\Omega$ is a subset of $\left(0, -\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) \subset \mathbb{R}$ such that $2\omega_{\downarrow} + \omega_{\downarrow}^2 < \omega_{\uparrow}$, and $\omega_{\downarrow}$ and $\omega_{\uparrow}$ are respectively the least upper and the greatest lower bounds on members of $\Omega$.

That all axioms of QSM, the mereology with crisp size predicates, are true in all $\Omega$-structures was shown in Sections 4.2 and 4.3. It remains to consider the axioms that include vague size predicates, i.e., (A14-A30). These axioms fall into four groups: (i) axioms (A14-A23, A25, A26) that are true for $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$, (ii) axiom (A24) that is true for $0 < 2\omega_{\downarrow} + \omega_{\downarrow}^2 \leq \omega_{\uparrow} < 1$, (iii) axioms (A27, A28) that are true for that $\omega_{\downarrow} < \omega_{\uparrow}/(1 + \omega_{\uparrow}) < \omega_{\uparrow}$, and (iv) axioms (A29, A30) that are true for that $0 < \omega < -\frac{1}{2} + \frac{1}{2}\sqrt{5}$.

#### 7.1. Axioms that are true for precisifications within the range $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$

Firstly, to rule out special cases in which $\mathcal{V}(\sim) = \mathcal{V}(\approx) = \emptyset$, and $\mathcal{V}(\lessapprox) = \emptyset$, $\Omega$ is restricted such that $0 < \omega_{\downarrow} \leq \omega \leq \omega_{\uparrow} < 1$ for all $\omega \in \Omega$ (Section 5.2). Consider the formula $(\exists y)\forall x \sim y$ which, clearly, is false in $\Omega$-structures in which $\Omega$ is a singleton set. Such cases are ruled out by restricting $\Omega$ such that $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$ and $\omega_{\downarrow} \leq \omega \leq \omega_{\uparrow}$ for all $\omega \in \Omega$.

**Lemma 4.** Axioms (A14 – A23, A25, A26) are true for all choices of $\Omega$ such that $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$ and $\omega_{\downarrow} \leq \omega \leq \omega_{\uparrow}$ for all $\omega \in \Omega$. 

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Consider Table 2 which summarizes important compositional theorems of VSM. (This table is not intended to serve as a composition table for automated reasoning purposes in the sense of (Renz and Nebel, 1999) or (Eschenbach, 2001).) The top left quadrant of the interior the table contains theorems about the composition of crisp size predicates. These theorems belong to QSM, the crisp core of VSM (Section 4.3). The theorems in the top right and the bottom left quadrants are such that vague and crisp size predicates are composed. These theorems explicate logical properties of vague size predicates in relationship to crisp size predicates via their logical composition (Section 6.1).

The bottom right quadrant of Table 2 lists theorems about the composition of vague size predicates. There are empty slots in this quadrant indicating that corresponding compositional sentences cannot be axioms or theorems of VSM. This primarily affects compositional sentences that include the vague predicates $\approx$ and $\lessapprox$. The inclusion of any of those sentences would give rise to inconsistencies, because the positive extension of $\approx$ serves as relation of ‘irrelevant difference’ in $F$-Sorites series. The table also shows that vague binary predicates such as $\lessapprox$ and $\lesslessapprox$ can be composed without giving rise to contradictions. Consequently, it is important to distinguish vague predicates that serve as relation of ‘irrelevant difference’ in $F$-Sorites series from vague predicates that do not.

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Important compositional theorems of VSM.

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For the axioms of the theory VSM to be true in $\Omega$-structures, the precisification points in $\Omega$ need to be within logically admissible ranges. In this section it is shown that the axioms of VSM are true in $\Omega$-structures, $(\Omega, \mathcal{D}, \mathcal{V}, \subseteq, \|\|)$, where $\Omega$ is a subset of $\left(0, -\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) \subset \mathbb{R}$ such that $2\omega_{\downarrow} + \omega_{\downarrow}^2 < \omega_{\uparrow}$, and $\omega_{\downarrow}$ and $\omega_{\uparrow}$ are respectively the least upper and the greatest lower bounds on members of $\Omega$.

That all axioms of QSM, the mereology with crisp size predicates, are true in all $\Omega$-structures was shown in Sections 4.2 and 4.3. It remains to consider the axioms that include vague size predicates, i.e., (A14-A30). These axioms fall into four groups: (i) axioms (A14-A23, A25, A26) that are true for $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$, (ii) axiom (A24) that is true for $0 < 2\omega_{\downarrow} + \omega_{\downarrow}^2 \leq \omega_{\uparrow} < 1$, (iii) axioms (A27, A28) that are true for that $\omega_{\downarrow} < \omega_{\uparrow}/(1 + \omega_{\uparrow}) < \omega_{\uparrow}$, and (iv) axioms (A29, A30) that are true for that $0 < \omega < -\frac{1}{2} + \frac{1}{2}\sqrt{5}$.

#### 7.1. Axioms that are true for precisifications within the range $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$

Firstly, to rule out special cases in which $\mathcal{V}(\sim) = \mathcal{V}(\approx) = \emptyset$, and $\mathcal{V}(\lessapprox) = \emptyset$, $\Omega$ is restricted such that $0 < \omega_{\downarrow} \leq \omega \leq \omega_{\uparrow} < 1$ for all $\omega \in \Omega$ (Section 5.2). Consider the formula $(\exists y)\forall x \sim y$ which, clearly, is false in $\Omega$-structures in which $\Omega$ is a singleton set. Such cases are ruled out by restricting $\Omega$ such that $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$ and $\omega_{\downarrow} \leq \omega \leq \omega_{\uparrow}$ for all $\omega \in \Omega$.

**Lemma 4.** Axioms (A14 – A23, A25, A26) are true for all choices of $\Omega$ such that $0 < \omega_{\downarrow} < \omega_{\uparrow} < 1$ and $\omega_{\downarrow} \leq \omega \leq \omega_{\uparrow}$ for all $\omega \in \Omega$. 

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Proof. Axioms (A14) and (A15) are obviously true. Axiom (A16) is true since \( D \) is closed under finite unions (A5) and, on the intended interpretation, members of \( D \) are guaranteed to have a finite size.

Consider Axiom (A17). To show \( V((\exists y)(U(x \approx y) \land \neg(x \sim y))) = 1 \). By Defs 9, 10 and Lemma 1, one has to show that there exists a \( y \in D \) such that \( 1/(1 + \omega_1) \leq |x||/|y|| \leq 1 + \omega_1 \) and \( |x||/|y|| \neq 1 \). This is the case if \( 0 < \omega_1 < 1 \).

Consider (A18) and assume that \( V(x \approx y \land x \leq z \land z \leq y, \omega) = 1 \). By Definitions 9 and 10: \( 1/(1 + \omega) \leq |x||/|y|| \leq 1 + \omega \) and \( |x||/|z|| \leq |y||/|z|| \leq 1 + \omega \). Thus \( 1/(1 + \omega) \leq |x||/|z|| \leq 1 + \omega \) and \( 1/(1 + \omega) \leq |x||/|z|| \leq 1 + \omega \). That is, \( V(z \approx x \land z \approx y, \omega) = 1 \). Hence, \( V(x \approx y \land x \leq z \land z \leq y, \omega) = 1 \).

Consider (A19). \( V(x \ll y \land y \leq z, \omega) = 1 \) if and only if \( |x||/|y|| < \omega/(1 + \omega) \) and \( |y||/|z|| \leq |x||/|z|| \). Thus, \( |x||/|z|| < \omega/(1 + \omega) \). That is, \( V(x \ll z, \omega) = 1 \). Hence, \( V(x \ll y \land y \leq z, \omega) = 1 \).

Consider (A20). To show \( V((\exists y)(x \ll y))) = 1 \), i.e., \( V((\exists y)(S(x \ll y) \land S^-(x \ll y))) = 1 \), i.e., there exists a \( e \in D \) such that \( 1/(1 + \omega) < |x||/|d|| \leq 1/(1 + \omega) \). Thus, by Lemma 2 and Def. 13, \( V(S(x \ll d)) = 1 \).

Consider (A21). To show \( V((\exists y)(x \ll y))) = 1 \), i.e., \( V((\exists y)(S(x \ll y) \land S^-(x \ll y))) = 1 \), i.e., there exists a \( y \in D \) such that \( S(x \ll y) = 1 \) and \( V(U(x \ll d)) = 0 \). Since \( 0 < \omega_1 < \omega_1 < 1 \), there is a \( d \in D \) such that \( 1/(1 + \omega) < |x||/|d|| \leq 1/(1 + \omega) \). Thus, \( V(S(x \ll d) = 1 \).

Consider (A22). Assume \( V(S(x \ll y), \omega) = 1 \). By Def. 13 and Lemma 2: \( |x||/|y|| < \omega/(1 + \omega) \).

Assume that \( 0 < \omega_1 < \omega_1 < 1 \). Then \( \omega/(1 + \omega) < 1/(1 + \omega) \). Thus, \( |x||/|y|| < 1/(1 + \omega) \). By Lemma 2 and Def. 13: \( V(U(x \ll y), \omega) = 1 \).

Consider (A23). Assume that \( V(U(x \approx y) \land \neg(x \sim y), \omega) = 1 \). By Lemma 1 and Def. 9, \( V(x \approx y, \omega) = 1 \) and \( V(x \sim y, \omega) = 0 \). By Def. 10, either (i) \( 1/(1 + \omega_1) \leq |x||/|y|| \leq 1 + \omega_1 \) and consider (i) and let \( z \in D \) such that \( 1/(1 + \omega) = |x||/|y|| \). Algebraic computations show that \( 1/(1 + \omega) \leq |z||/|y|| \leq 1/(1 + \omega) \). Similarly, consider (ii) and let \( z \in D \) such that \( |z||/|y|| = 1 + \omega_1 \). Then, algebraic computations show that \( 1 + \omega_1 \leq |z||/|y|| \leq 1 + \omega_1 \). Hence, in both cases, (i) and (ii), there exists a \( z \in D \) such that \( 1 + \omega_1 \leq |z||/|y|| \leq 1 + \omega_1 \) and it is not the case that \( 1/(1 + \omega_1) \leq |z||/|y|| \leq 1/(1 + \omega_1) \). Thus, by Def. 10, \( V(z \approx x, \omega) = 1 \) and \( V(z \approx y, \omega) = 0 \). By Lemma 1, \( V(U(z \approx x, \omega) = 1 \) and \( V(U(z \approx y, \omega) = 0 \). By Def. 9, \( V(U(z \approx y, \omega) = 1 \). Hence, \( V(U(x \approx y) \land \neg(x \sim y)) = 1 \).

That (A25) and (A26) are true immediately follows from Lemma 3 (Equations 17 and 18).

7.2. Logically admissible precisification ranges for the weak transitivity axiom (A24)

The weak transitivity axiom for \( \approx \) (A24) is not true for all choices of \( \Omega \) with \( 0 < \omega_1 < \omega_1 < 1 \). Assume \( d_1, d_2, \) and \( d_3 \) such that \( (d_1, d_2) \) and \( (d_2, d_3) \) are members of the positive extension of the predicate \( \approx \) but their differences in Lebesgue measures are maximal, i.e., \( 1/(1 + \omega_1) = |d_1||/|d_2|| = |d_2||/|d_3|| \). For axiom (A24) to be true, the difference \( \omega_1 - \omega_1 \) has to be such that the combined differences in Lebesgue measures of \( (d_1, d_2) \) and \( (d_2, d_3) \), that characterize \( (d_1, d_3) \), are smaller than or equal to \( 1/(1 + \omega_1) \):

**Lemma 5.** Axiom (A24) is true for all choices of \( \Omega \) such that \( 0 < 2\omega_1 + \omega_1^2 \leq \omega_1 \leq 1 \) and \( \omega_1 \leq \omega \leq \omega_1 \) for all \( \omega \in \Omega \).

Proof. Assume that \( V(U(x \approx y) \land U(y \approx z), \omega) = 1 \). By Lemma 1: \( V((x \approx y), \omega') = 1 \) and \( V((y \approx z), \omega) = 1 \). By Def. 10: \( 1/(1 + \omega_1) \leq |x||/|y|| \leq (1 + \omega_1) \) and \( 1/(1 + \omega_1) \leq |y||/|z|| \leq (1 + \omega_1) \). Thus, \( 1/(1 + \omega_1) \leq |x||/|z|| \leq (1 + \omega_1) \). Assume that \( 0 < 2\omega_1 + \omega_1^2 \leq \omega_1 \leq 1 \). Then \( 1/(1 + \omega_1) \leq |x||/|z|| \leq (1 + \omega_1) \). Thus, \( 1/(1 + \omega) \leq |x||/|z|| \leq (1 + \omega) \). By Def. 10: \( V(x \approx z, \omega) = 1 \). By Lemma 1: \( V(S(x \approx z), \omega) = 1 \). Hence, \( V(U(x \approx y) \land U(y \approx z) = 1 \).
7.3. Logically admissible precisification ranges for the composition axioms (A27–A30)

Axioms (A27) and (A28) are not true for all choices of \(\Omega\) with \(0 < \omega_\downarrow < \omega_\uparrow < 1\). Consider (A27) and assume domain members \(d_1, d_2, \) and \(d_3\) and a precisification \(\omega \in \Omega\), such that \(\langle d_1, d_2, \omega \rangle\) is a member of the extension of the predicate \(\ll\) and \(\langle d_2, d_3, \omega \rangle\) is a member of the extension of the predicate \(\approx\). Assume further that \(d_1\) is fixed and that \(d_2\) is such that the differences in Lebesgue measures between \(d_1\) and \(d_2\) is maximal for the precisification \(\omega_\uparrow\). In addition, assume that \(d_3\) is such that the differences in Lebesgue measures between \(d_2\) and \(d_3\) is maximal for the precisification \(\omega_\downarrow\). For axiom (A27) to be true, \(\langle d_1, d_3 \rangle\) is a member of the complement of the negative extension of the predicate \(\ll\). That is, the ratio of the Lebesgue measures of the domain members \(d_1\) and \(d_3\) must be smaller than \(\omega_\uparrow/(1 + \omega_\uparrow)\). This is the case if \(\omega_\downarrow < \omega_\uparrow/(1 + \omega_\uparrow)\):

**Lemma 6.** Axioms (A27) and (A28) are true for all choices of \(\Omega\) such that \(0 < \omega_\downarrow < \omega_\uparrow/(1 + \omega_\uparrow) < \omega_\uparrow < 1\).

**Proof.** Consider axiom (A27) and assume that \(V(U(x \ll y) \land U(y \approx z), \omega) = 1\). By Defs. 10 and 13: \(|x||/|y|| < \omega_\downarrow/(1 + \omega_\downarrow)\) and \(1/(1 + \omega_\downarrow) \leq |y||/|z|| \leq 1 + \omega_\downarrow\). It follows: \(|x||/|z|| < \omega_\downarrow/(1 + \omega_\downarrow)\). Assume that \(\omega_\downarrow < \omega_\uparrow/(1 + \omega_\uparrow)\). Then \(|x||/|z|| < \omega_\downarrow < \omega_\uparrow/(1 + \omega_\uparrow)\). By Def. 13 and Lemma 2: \(V(S(x \ll z), \omega) = 1\).

Consider axiom (A28) and assume that \(V(U(x \approx y) \land U(y \ll z), \omega) = 1\). By Defs. 10 and 13: \(1/(1 + \omega_\downarrow) \leq |x||/|y|| \leq 1 + \omega_\downarrow\) and \(|y||/|z|| < \omega_\downarrow/(1 + \omega_\downarrow)\). It follows: \(|x||/|z|| < \omega_\downarrow/(1 + \omega_\downarrow)\). Assume that \(\omega_\downarrow < \omega_\uparrow/(1 + \omega_\uparrow)\). Then \(|x||/|z|| < \omega_\downarrow < \omega_\uparrow/(1 + \omega_\uparrow)\). By Def. 13 and Lemma 2: \(V(S(x \ll z), \omega) = 1\).

Hence, \(V(U(x \approx y) \land U(y \ll z) \rightarrow S(x \ll z)) = 1\).

Finally consider Axioms (A29) and (A30). As in the previous cases, (A29) and (A30) are not true for all choices of \(\Omega\) with \(0 < \omega_\downarrow < \omega_\uparrow < 1\). Using algebraic computations similarly to those used in the previous Lemmata one can show:

**Lemma 7.** Axioms (A29) and (A30) are true for all choices of \(\Omega\) such that \(0 < \omega < -\frac{1}{2} + \frac{1}{2} \sqrt{5}\).

**Proof.** Consider (A29) and assume \(V(x \ll y \land y \approx z, \omega) = 1\). By Defs. 10 and 13: \(|x||/|y|| < \omega/(1 + \omega)\) and \(1/(1 + \omega) \leq |y||/|z|| \leq 1 + \omega\). It follows: \(|x||/|z|| < \omega/(1 + \omega) < \omega/(1 + \omega)^2 < \omega\). To ensure that \(\omega < 1/(1 + \omega)\), assume that \(0 < \omega < -\frac{1}{2} + \frac{1}{2} \sqrt{5}\). The golden ratio conjugate \(-\frac{1}{2} + \frac{1}{2} \sqrt{5}\) is such that \(\omega = 1/(1 + \omega)\). Thus, \(|x||/|z|| < \omega/(1 + \omega)^2 < \omega < 1/(1 + \omega)\). That is, \(|x||/|z|| < 1/(1 + \omega)\). By Def. 13: \(V(x \ll z, \omega) = 1\).

Hence, \(V(x \ll y \land y \approx z \rightarrow x \ll z) = 1\). Similarly for axiom (A30).

7.4. Precisifications that are admitted by all axioms of VSM

Every \(\Omega\)-structure has a fixed precisification range \(\Omega\) with the greatest lower bound \(\omega_\downarrow\) and the least upper bound \(\omega_\uparrow\), i.e., \(\omega_\downarrow \leq \omega \leq \omega_\uparrow\) for all \(\omega \in \Omega\). The logically admissible precisifications in \(\Omega\) are such that trivial cases are excluded. In addition, precisification parameters need to be such that all axioms of VSM are satisfied. As detailed above, these axioms fall into four groups: (i) axioms (A14-A23, A25, A26) that are true for \(0 < \omega_\downarrow < \omega_\uparrow < 1\), (ii) axiom (A24) that is true for \(0 < 2\omega_\downarrow + \omega_\downarrow^2 \leq \omega_\uparrow < 1\), (iii) axioms (A27, A28) that are true for that \(\omega_\downarrow < \omega_\uparrow/(1 + \omega_\uparrow) < \omega_\uparrow\), and (iv) axioms (A29, A30) that are true for that \(0 < \omega < -\frac{1}{2} + \frac{1}{2} \sqrt{5}\). For all constraints to be satisfied, the conditions of Theorem 1 need to be met:

**Theorem 1.** The axioms (A14-A30) of VSM are true in \(\Omega\)-structures, \(\langle \Omega, D, V, \ll, |||| \rangle\), where \(\Omega \subseteq [\omega_\downarrow, \omega_\uparrow] \subset (0, -\frac{1}{2} + \frac{1}{2} \sqrt{5}) \subset \mathbb{R}\) such that \(2\omega_\downarrow + \omega_\downarrow^2 < \omega_\uparrow\).
follows that
\[ 0 < \omega < - \omega^{-1}. \]
By Lemmata 5 and 6:
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Proof. By Lemmata 5 and 6: \( -\frac{\omega}{\omega^{-1}} \) \( < \omega \) and \( 2\omega + \omega^2 \leq \omega \). Thus, either (a) \( -\frac{\omega}{\omega^{-1}} \leq 2\omega + \omega^2 \) or (b) \( 2\omega + \omega^2 \leq -\frac{\omega}{\omega^{-1}} \). Algebraic computations together with the constraints of Lemma 4 yield that (a) is the case if \( -\frac{1}{2} + \frac{1}{2} \sqrt{5} \leq \omega \) \( < 1 \) and that (b) is the case if \( 0 \leq \omega \leq -\frac{1}{2} + \frac{1}{2} \sqrt{5} \). Since, by Lemma 7, \( 0 < \omega < -\frac{1}{2} + \frac{1}{2} \sqrt{5}, (a) \) is ruled out and thus \( \omega \leq 2\omega + \omega^2 \leq -\frac{\omega}{\omega^{-1}} \leq \omega \leq -\frac{1}{2} + \frac{1}{2} \sqrt{5} \). It then follows that \( [\omega, \omega] \subseteq (0, -\frac{1}{2} + \frac{1}{2} \sqrt{5}) \subseteq \mathbb{R} \) such that \( 2\omega + \omega^2 < \omega \).

Thus, the logically admissible precisifications in \( \Omega \) must not exceed a maximal degree of coarseness \( (\omega < -\frac{1}{2} + \frac{1}{2} \sqrt{5}) \) and they need to have precisification ranges satisfying certain constraints regarding their extension \( (2\omega + \omega^2 < \omega) \). Theorem 1 also shows that VSM is consistent. Note that \( \Omega \)-structures here are used as examples to illustrate a plausible class of models for VSM. For this reason, no attempt is made to provide a complete axiomatization for such structures.

8. Conclusions

In this paper logical and semantical properties of vague size predicates were formalized in the theory VSM – a mereology of regions extended by crisp and vague size predicates. VSM was formulated in the modal language LV, in which, in addition to the logical properties and interrelations, also semantic distinctions between crisp and vague size predicates can be made explicit.

The restriction of the scope of the formal theory to vague size predicates enabled the use of a rather simple parametrized linear constraints to interpret vague size predicates in set-theoretic structures (\( \Omega \)-structures). The use of such simple linear constraints may be too simple to capture all aspects of the semantics of vague size predicates. However, there do seem to be good reasons to agree on the following:

(a) The logical properties and interrelations of size predicates as they are specified by the axioms of VSM;
(b) The semantic properties of vague size predicates as they are specified by the axioms of VSM;
(c) The specification of the interpretation of vague size predicates in terms of parametrized constraints (of whatever form) on the ratio of measures in a certain class of measure spaces.

According to the underlying supervaluation semantics, admissible precisifications of vague size predicates are represented in \( \Omega \)-structures by precisification parameters that modify the underlying constraint-based models. Structures in which every model has a fixed set of admissible precisifications cannot model all aspects of the semantics of vague size predicates. This is because such structures do not permit to take into account higher order vagueness and context dependency:

Higher order vagueness. On the semantic account of vagueness this means that there are many equally good ways to precisify the boundaries of the positive as well as the negative extension of a vague predicate such as \( \approx \). On the interpretation specified in Equations 9 and 13, the boundaries of the positive and the negative extensions of the vague size predicates are crisp and determined by the lower and upper bounds of the underlying precisifications in \( \Omega \). That such boundaries are subject to vagueness themselves, can be taken into account in models with multiple precisification ranges. In such models, each way of precisifying the positive and the negative extensions of a vague size predicate, can be understood as a different choice of \( \Omega \). In (Bittner, 2011a) additional modal operators are introduced whose interpretation depends on precisifications of extensions of vague size predicates.

As mentioned in Section 2, vague predicates such as "has normal systolic blood pressure", "has optimal cholesterol level" are context-dependent in the sense that they have different interpretations in different contexts. Similarly for the more general vague size predicates "has roughly the same size as" and "is negligible in size with respect to"; There may be contexts in which a certain precisification range \( \Omega \) will be more suitable for the interpretation of these predicates than others. Thus, higher order vagueness as well as context dependency can be analyzed by allowing for multiple ways of precisifying the positive and the negative extensions of vague size predicates (Bittner, 2011a).
The main focus of this paper was on logical and general semantic properties of vague size predicates. Such properties are independent of specific precisification ranges as long as they are logically admissible in the sense of Theorem 1. That is, such properties are not affected by higher order vagueness and context dependency.

References


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